



HARMONIC NUMBERS AND THE PRIME COUNTING FUNCTION

Jesse Elliott

*Department of Mathematics, California State University, Channel Islands,
Camarillo, California*
jesse.elliott@csuci.edu

Received: 8/9/20, Revised: 1/3/21, Accepted: 3/1/21, Published: 3/23/21

Abstract

We provide approximations to the prime counting function by various discretized versions of the logarithmic integral function, expressed solely in terms of the harmonic numbers. We demonstrate with explicit error bounds that these approximations are at least as good as the logarithmic integral approximation. As a corollary, we provide some reformulations of the Riemann hypothesis in terms of the prime counting function and the harmonic numbers.

1. Introduction

This paper concerns the function $\pi : \mathbb{R}_{>0} \rightarrow \mathbb{R}$ that for any $x > 0$ counts the number of primes less than or equal to x :

$$\pi(x) = \#\{p \leq x : p \text{ is prime}\}, \quad x > 0.$$

The function $\pi(x)$ is known as the *prime counting function*. We call the related function $\mathbf{p} : \mathbb{R}_{>0} \rightarrow \mathbb{R}$ defined by

$$\mathbf{p}(x) = \frac{\pi(x)}{x}, \quad x > 0,$$

the *prime density function*. The celebrated *prime number theorem*, proved independently by de la Vallée Poussin [2] and Hadamard [7] in 1896, states that

$$\pi(x) \sim \frac{x}{\log x} \quad (x \rightarrow \infty),$$

where $\log x$ is the natural logarithm. It is known, however, that the *logarithmic integral function*

$$\text{li}(x) = \int_0^x \frac{dt}{\log t}, \quad x > 0,$$

(where the integral assumes the Cauchy principal value and $\text{li}(1) = -\infty$), provides a better approximation to $\pi(x)$ than any algebraic function of $\log x$. The *prime number theorem with error term*, proved by de la Vallée Poussin in 1899 [3], states that

$$\pi(x) - \text{li}(x) = O\left(xe^{-C\sqrt{\log x}}\right) \quad (x \rightarrow \infty)$$

for some constant $C > 0$. De la Vallée Poussin’s result has since been improved to

$$\pi(x) - \text{li}(x) = O\left(xe^{-A(\log x)^{3/5}(\log \log x)^{-1/5}}\right) \quad (x \rightarrow \infty),$$

where $A = 0.2098$ [5], which is the strongest known O bound on $\pi(x) - \text{li}(x)$ to date.

Proofs of the strongest known bounds on the error $\pi(x) - \text{li}(x)$ are based on Riemann’s explicit formula for $\pi(x)$ in terms of the zeros of the Riemann zeta function $\zeta(s)$ and advanced methods for verifying zero-free regions of $\zeta(s)$ in the critical strip $0 \leq \text{Re } s \leq 1$. The celebrated *Riemann hypothesis* states that all such zeros lie on the line $\text{Re } s = \frac{1}{2}$. As is now well known, von Koch proved in 1901 [8] that the Riemann hypothesis is equivalent to

$$\pi(x) - \text{li}(x) = O(\sqrt{x} \log x) \quad (x \rightarrow \infty).$$

It is known, more generally, that if

$$\delta = \sup\{\text{Re}(s) : s \in \mathbb{C}, \zeta(s) = 0\}$$

denotes the supremum of the real parts of the zeros of $\zeta(s)$, then $\frac{1}{2} \leq \delta \leq 1$, and δ is the least $\alpha \in \mathbb{R}$ such that

$$\pi(x) - \text{li}(x) = O(x^\alpha \log x) \quad (x \rightarrow \infty). \tag{1.1}$$

(See, for example, [12, Theorem 15.2 and Section 13.1.1 Exercise 1].) Moreover, the Riemann hypothesis is equivalent to $\delta = \frac{1}{2}$.

For every positive integer n , let $H_n = \sum_{k=1}^n \frac{1}{k}$ denote the n th *harmonic number*. The *summatory function* of a function $f(x)$ is the function $\sum_{k=1}^n f(k)$. Thus, the function H_n is the summatory function of $f(x) = \frac{1}{x}$. Summatory functions are *discrete integrals* in the sense that $\sum_{k=N}^n f(k) = \int_N^n f(x) d\nu(x)$ for all integers $n \geq N$, where ν is the unique discrete measure with respect to Lebesgue measure that is supported on \mathbb{Z} with all weights equal to 1. Thus, the n th harmonic number $H_n = \int_1^n \frac{1}{x} d\nu(x)$ is a discrete integral of $\frac{1}{x}$ and is in this sense a “discrete natural logarithm.” Not unexpectedly, one has

$$H_n \sim \int_1^n \frac{dx}{x} = \log n \quad (n \rightarrow \infty),$$

and, more precisely, the limit

$$\lim_{n \rightarrow \infty} (H_n - \log n) = \gamma = 0.577215664901\dots,$$

known as the *Euler–Mascheroni constant*, is finite, and represents a precise measure of the discrepancy between the natural logarithm and the “discrete natural logarithm.”

Because $H_n \sim \log n$ ($n \rightarrow \infty$), the prime number theorem is equivalent to

$$\pi(n) \sim \frac{n}{H_n} \quad (n \rightarrow \infty),$$

where $\frac{n}{H_n}$ is also the harmonic mean of the integers $1, 2, 3, \dots, n$. This simple observation, alongside an inequality equivalent to the Riemann hypothesis involving the sum of divisors function and the harmonic numbers discovered by J. Lagarias, described below, led us to wonder if the harmonic numbers could be used to provide approximations to $\pi(n)$ that are better than $\frac{n}{H_n}$ —or, ideally, even as good as $\text{li}(n)$. The former problem in part inspired the paper [4], where we provide various asymptotic expansions of the prime counting function, including several involving the harmonic numbers, such as the (divergent) asymptotic continued fraction expansion

$$\mathbf{p}(e^\gamma n) \sim \frac{1/H_n}{1 - \frac{1/H_n}{1 - \frac{1/H_n}{1 - \frac{2/H_n}{1 - \frac{2/H_n}{1 - \frac{3/H_n}{1 - \frac{3/H_n}{1 - \dots}}}}}}} \quad (n \rightarrow \infty)$$

of $\mathbf{p}(e^\gamma n)$ [4, Corollary 4.5], where

$$e^\gamma = \lim_{n \rightarrow \infty} \frac{e^{H_n}}{n} = 1.781072417990\dots,$$

and also where

$$e^\gamma = \lim_{x \rightarrow \infty} \frac{\mathbf{p}(x)}{\prod_{p \leq x} \left(1 - \frac{1}{p}\right)}$$

due to the prime number theorem and the third of Mertens’ famous three theorems of 1874 [10]. In 1984, G. Robin proved [13] that the Riemann hypothesis holds if and only if

$$\sum_{d|n} d \leq e^\gamma n \log \log n, \quad \forall n \geq 5041.$$

Since by Mertens’ second theorem and a 1913 result of Gronwall [6] one also has

$$e^\gamma = \limsup_{n \rightarrow \infty} \frac{\sum_{d|n} d}{n \log \log n} = \limsup_{n \rightarrow \infty} \frac{\sum_{d|n} \frac{1}{d}}{\sum_{p \leq n} \frac{1}{p}},$$

the constant e^γ in Robin's equivalence is the best possible. In 2000, J. Lagarias used Robin's result to show [9] that the Riemann hypothesis holds if and only if

$$\sum_{d|n} d < H_n + e^{H_n} \log H_n, \quad \forall n > 1,$$

if and only if

$$\sum_{d|n} d < e^{H_n} \log H_n, \quad \forall n > 60.$$

Lagarias' inequalities are closely related to Robin's because, by asymptotics noted earlier, one has

$$e^\gamma n \log \log n \sim e^{H_n} \log H_n \sim H_n + e^{H_n} \log H_n \quad (n \rightarrow \infty).$$

The three "elementary" reformulations of the Riemann hypothesis noted above concern the sum of divisors function rather than the prime counting function. In this paper, we provide several reformulations of the Riemann hypothesis that are expressed solely in terms of the harmonic numbers and the prime counting function. For example, we show in Section 5 that the Riemann hypothesis holds if and only if

$$\mathbf{p}(e^\gamma n) = \frac{1}{n} \sum_{k=1}^n \frac{1}{H_k} + O\left(\frac{H_n}{\sqrt{n}}\right) \quad (n \rightarrow \infty),$$

if and only if

$$\left| \mathbf{p}(e^\gamma n) - \frac{1}{n} \sum_{k=2}^{n-1} \frac{1}{H_k} \right| < \frac{1}{8\pi e^{\gamma/2}} \frac{H_n}{\sqrt{n}} + \frac{1 + 0.4986013304}{n}, \quad \forall n \geq 1.$$

Moreover, any choice of larger constants still yields a Riemann hypothesis equivalent, so, for example, since $\frac{1}{8\pi e^{\gamma/2}} = \frac{1}{33.541358\dots} < \frac{1}{33}$, the Riemann hypothesis is also equivalent to

$$\left| \mathbf{p}(e^\gamma n) - \frac{1}{n} \sum_{k=2}^{n-1} \frac{1}{H_k} \right| < \frac{1}{33} \frac{H_n}{\sqrt{n}} + \frac{3}{2n}, \quad \forall n \geq 1.$$

The second and third of our Riemann hypothesis equivalents above are made possible by a well-known reformulation due to L. Schoenfeld [14]: the Riemann hypothesis holds if and only if

$$|\pi(x) - \text{li}(x)| < \frac{1}{8\pi} \sqrt{x} \log x, \quad \forall x \geq 2657. \tag{1.2}$$

The constant 0.4986013304 in the second Riemann hypothesis equivalent can be replaced with any other upper bound of the limit

$$\kappa = \lim_{n \rightarrow \infty} \left(\frac{\text{li}(e^\gamma n)}{e^\gamma} - \sum_{k=1}^{n-1} \frac{1}{H_k} \right) \approx 0.4986.$$

The limit κ exists because the sequence $\frac{\text{li}(e^\gamma n)}{e^\gamma} - \sum_{k=1}^{n-1} \frac{1}{H_k}$ is positive, strictly increasing, and bounded above, and therefore bounded above by κ . Our results allow us to compute tight upper and lower bounds of constants like κ , so that, for example, one has

$$0.4985987518 < \kappa < 0.4986013304.$$

Moreover, since $\text{li}(x) > \pi(x) + \frac{1.49}{2}$ for all $x \geq 6$ for which the value of $\pi(x)$ is known, the sum $\sum_{k=2}^{n-1} \frac{1}{H_k}$ is closer to $\frac{\pi(e^\gamma n)}{e^\gamma}$ than is $\frac{\text{li}(e^\gamma n)}{e^\gamma}$ for all $n \geq 6$ for which the value of $\pi(e^\gamma n)$ is known. Thus, approximations of the prime counting function using harmonic numbers can indeed be worthy rivals of the standard logarithmic integral approximation.

In Section 3, we prove that, for all $t \in \mathbb{R}$, one has

$$\begin{aligned} \frac{\text{li}(e^t n)}{e^t} &= \sum_{\mu e^{-t} \leq k < n} \frac{1}{H_k - \gamma + t} + \beta_n(t) \\ &= \sum_{\mu e^{-t} \leq k < n} \frac{1}{H_k - \gamma + t} + \beta(t) - \frac{1 + o(1)}{12n(\log n)^2} \quad (n \rightarrow \infty) \end{aligned}$$

for unique error functions $\beta_n(t) > 0$ and $\beta(t) > 0$ with $\beta(t) = \lim_{n \rightarrow \infty} \beta_n(t)$, where $\mu = 1.451369234883\dots$ is the unique positive zero of $\text{li}(x)$, called the *Ramanujan–Soldner constant*. We also prove explicit bounds on $\beta_n(t)$ and $\beta(t)$ in terms of t that allow us to compute $\beta(t)$ to any desired degree of accuracy, and we show, for example, that

$$\begin{aligned} \beta(t) &< \limsup_{t \rightarrow -\infty} \beta(t) = \frac{1}{\log \mu} = 2.684510350820\dots, \\ \beta(t) &= \frac{\text{li}(e^t \lceil \mu e^{-t} \rceil)}{e^t} + \frac{(1 + o(1))e^t}{12\mu(\log \mu)^2} \quad (t \rightarrow -\infty), \end{aligned}$$

and

$$\beta(t) = \frac{1}{t} + O\left(\frac{1}{t^2}\right) \quad (t \rightarrow \infty),$$

where $\lceil x \rceil$ (resp., $\lfloor x \rfloor$) denotes the ceiling (resp., floor) of x for any real number x . Note that the constant $\kappa \approx 0.4986$ introduced earlier is precisely $\beta(\gamma)$.

In Section 4, we use the results noted above to make precise the approximation

$$\frac{\pi(e^t n)}{e^t} \approx \sum_{\mu e^{-t} \leq k < n} \frac{1}{H_k - \gamma + t}, \quad n \geq \mu e^{-t},$$

from which we derive our Riemann hypothesis equivalents in Section 5. Analogous to the integral representation $\int_{\mu e^{-t}}^x \frac{du}{t + \log u}$ of $\frac{\text{li}(e^t x)}{e^t}$, the sum $\sum_{\mu e^{-t} \leq k < x} \frac{1}{H_k - \gamma + t}$ can be represented as the discrete integral $\int_{\mu e^{-t}}^{x+0^-} \frac{d\nu(u)}{H_u - \gamma + t}$. Our approximation to $\frac{\pi(e^t n)}{e^t}$ above is therefore a doubly discretized version of the logarithmic integral.

Since the Riemann hypothesis, for all we currently know, could be false, we find it useful to generalize our reformulations of the hypothesis to unconditional results expressed in terms of the supremum of the real parts of the zeros of the Riemann zeta function. Equation (1.1) is the quintessential example of such a generalization. In fact, all of the “heavy lifting” regarding the prime counting function in this paper is accomplished by Equations (1.1) and (1.2) and a theorem of Montgomery and Vaughan (Theorem 3). We are thus able to focus most of our attention on using the harmonic numbers to approximate the logarithmic integral using elementary analysis.

I would like to thank Sean Lubner and Daniel Brice for writing Python code to check the inequalities in Corollaries 9 through 14 for small values of n .

2. Approximating $\log x$ With Harmonic Numbers

In this section, we list some properties of the harmonic numbers that form the basis for our results.

From the functional equation

$$\Gamma(z + 1) = z\Gamma(z), \quad z \in \mathbb{C} \setminus \{0, -1, -2, -3, \dots\}$$

for the gamma function $\Gamma(z)$ follows, by logarithmic differentiation, the functional equation

$$\Psi(z + 1) = \frac{1}{z} + \Psi(z), \quad z \in \mathbb{C} \setminus \{0, -1, -2, -3, \dots\}$$

for the *digamma function* $\Psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$. Since $\Psi(1) = \Gamma'(1) = -\gamma$ and $H_0 = 0$, it follows that the harmonic numbers H_n are interpolated by the complex function

$$H_z = \Psi(z + 1) + \gamma = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{z + k} \right) = \lim_{n \rightarrow \infty} \left(H_n - \sum_{k=1}^n \frac{1}{z + k} \right), \quad z \in \mathbb{C} \setminus \{-1, -2, -3, \dots\}. \quad (2.1)$$

It is known that

$$H_z - \gamma - \log z \sim \frac{1}{2z} \quad (z \rightarrow \infty),$$

and, more generally, by the Euler-Maclaurin formula, that one has the (divergent) asymptotic expansion

$$H_z - \gamma - \log z - \frac{1}{2z} \sim \sum_{k=1}^{\infty} \frac{-B_{2k}}{2kz^{2k}} \quad (z \rightarrow \infty),$$

where B_k is the k th Bernoulli number. From this well-known expansion follows the asymptotic expansion

$$H_{z-1/2} - \gamma - \log z \sim \sum_{k=1}^{\infty} \frac{(1 - 2^{-2k+1})B_{2k}}{2kz^{2k}} \quad (z \rightarrow \infty).$$

From the latter expansion and [1, Theorem 8], one can show that

$$\sum_{k=1}^{n+1} \frac{(1 - 2^{-2k+1})B_{2k}}{2kx^{2k}} < H_{x-1/2} - \gamma - \log x < \sum_{k=1}^n \frac{(1 - 2^{-2k+1})B_{2k}}{2kx^{2k}}$$

for all $x > 0$ and all odd positive integers n . Thus, for example, one has

$$\frac{1}{24(x + 1/2)^2} - \frac{7}{960(x + 1/2)^4} < H_x - \gamma - \log(x + 1/2) < \frac{1}{24(x + 1/2)^2}, \quad \forall x > -1/2.$$

and therefore

$$\frac{1}{24(x + 1)^2} < H_x - \gamma - \log(x + 1/2) < \frac{1}{24(x + 1/2)^2}, \quad \forall x > -1/2. \tag{2.2}$$

In particular, $\log(x + 1/2) + \gamma$ is an excellent approximation for H_x , and, correspondingly, $H_{x-1/2} - \gamma$ is an excellent approximation for $\log x$. Since

$$H_x - \gamma - \log x \sim \frac{1}{2x} \quad (x \rightarrow \infty),$$

while

$$H_x - \gamma - \log(x + 1/2) \sim \frac{1}{24x^2} \quad (x \rightarrow \infty),$$

the advantage gained by shifting the log by $1/2$ is clear. This makes sense heuristically because of the advantage, for monotonic functions, of the midpoint rule over left-hand or right-hand Riemann sums. See Figures 1 and 2 for a graphical comparison of the approximations above.

By the functional equation

$$H_z = \frac{1}{z} + H_{z-1}, \quad z \in \mathbb{C} \setminus \{0, -1, -2, -3, \dots\}$$

for H_z , Equation (2.2) generalizes as follows.

Proposition 1. *For all $x \in \mathbb{R} \setminus \{-1, -2, -3, \dots\}$ and all integers $n > x - 1/2$, one has*

$$\frac{1}{24(x + n + 1)^2} \leq (H_x - \gamma) - \left(\log(x + n + 1/2) - \sum_{k=1}^n \frac{1}{x + k} \right) \leq \frac{1}{24(x + n + 1/2)^2}.$$

Consequently, for all $x \in \mathbb{R} \setminus \{-1, -2, -3, \dots\}$ one has

$$\Psi(x + 1) = H_x - \gamma = \log(x + n + 1/2) - \sum_{k=1}^n \frac{1}{x + k} + \frac{1 + o(1)}{24n^2} \quad (n \rightarrow \infty).$$

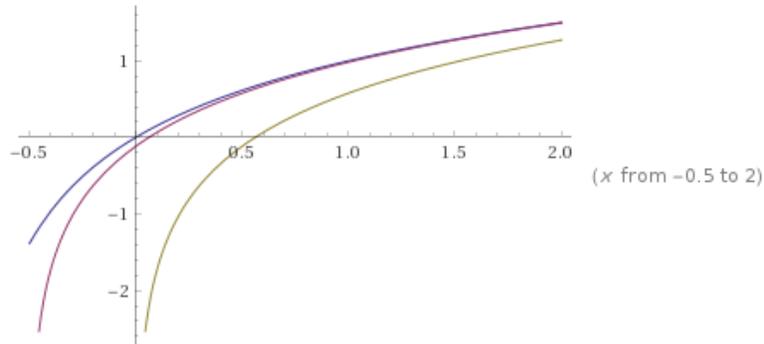


Figure 1: Graphs of $H_x > \log(x + 1/2) + \gamma > \log x + \gamma$ on $(-1/2, 2]$

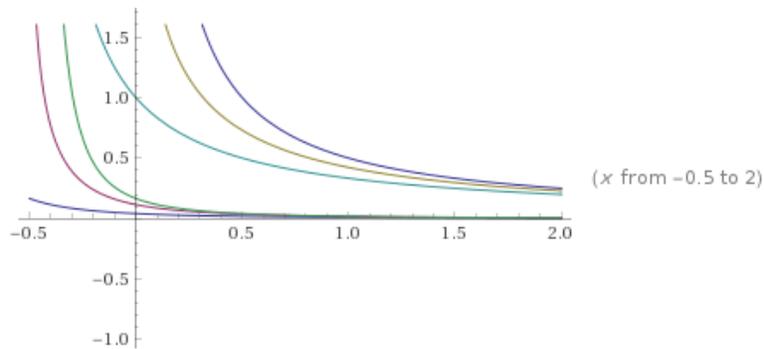


Figure 2: Graphs of $\frac{1}{24(x+1)^2}, H_x - \gamma - \log(x + 1/2), \frac{1}{24(x+1/2)^2}, \frac{1}{2(x+1/2)}, H_x - \gamma - \log x, \frac{1}{2x}$ on $(-1/2, 2]$, ordered from smallest to largest on $[0, \infty)$

3. Approximating $\text{li}(x)$ With Harmonic Numbers

Let $\mu = 1.451369234883\dots$ denote the *Ramanujan–Soldner constant*, which by definition is the unique positive zero of $\text{li}(x)$, or equivalently the unique positive real number μ such that $\text{li}(x) = \int_{\mu}^x \frac{dx}{\log x}$ for all $x > 1$. In this section we make precise the approximation

$$\frac{\text{li}(e^t n)}{e^t} \approx \sum_{k=\lceil \mu e^{-t} \rceil}^{n-1} \frac{1}{H_k - \gamma + t}, \quad \forall n \geq \lceil \mu e^{-t} \rceil.$$

More specifically, we prove the following.

Theorem 1. For all $t \in \mathbb{R}$, one has

$$\begin{aligned} \frac{\text{li}(e^t n)}{e^t} &= \sum_{k=\lceil \mu e^{-t} \rceil}^{n-1} \frac{1}{H_k - \gamma + t} + \beta_n(t) \\ &= \sum_{k=\lceil \mu e^{-t} \rceil}^{n-1} \frac{1}{H_k - \gamma + t} + \beta(t) - \frac{1 + o(1)}{12n(\log n)^2} \quad (n \rightarrow \infty) \end{aligned}$$

for unique error functions $\beta_n(t) > 0$ and $\beta(t) > 0$ with $\beta(t) = \lim_{n \rightarrow \infty} \beta_n(t)$.

To prove the theorem, we require the following notation.

Definition 1. Let N be a positive integer, and let $t \in \mathbb{R}$ with $t > -\log N$.

1. Let

$$\theta_n(t, N) = \int_N^n \frac{dx}{t + \log x} - \sum_{k=N}^{n-1} \frac{1}{H_k - \gamma + t}$$

for all $n \geq N$, and let

$$\theta(t, N) = \lim_{n \rightarrow \infty} \theta_n(t, N).$$

2. Let

$$\eta_n(t, N) = \sum_{k=N}^n \left(\frac{1}{t + \log(k + 1/2)} - \frac{1}{H_n - \gamma + t} \right).$$

for all $n \geq N$, and let

$$\eta(t, N) = \lim_{n \rightarrow \infty} \eta_n(t, N).$$

3. Let

$$\delta_n(t, N) = \int_N^n \frac{dx}{t + \log x} - \sum_{k=N}^{n-1} \frac{1}{\log(k + 1/2) + t}$$

for all $n \geq N$, and let

$$\delta(t, N) = \lim_{n \rightarrow \infty} \delta_n(t, N).$$

Our first goal is to prove the following theorem, from which we will then deduce Theorem 1.

Theorem 2. Let N be a positive integer, and let $t > -\log N$.

1. The sequence

$$\theta_n(t, N) = \int_N^n \frac{dx}{t + \log x} - \sum_{k=N}^{n-1} \frac{1}{H_k - \gamma + t}, \quad n > N$$

is positive, strictly increasing, and bounded above. Consequently, the limit

$$\theta(t, N) = \lim_{n \rightarrow \infty} \theta_n(t, N) > 0$$

exists.

2. For all $n \geq N$, one has

$$\begin{aligned} & \int_{n+1}^{\infty} \frac{dx}{24x^2(t + \log x)^2} + \frac{1}{24(n+1)(t + \log(n+1))^2} \\ & \leq \theta(t, N) - \theta_n(t, N) = \theta(t, n) \\ & \leq \int_n^{\infty} \frac{dx}{24x^2(t + \log x)^2} + \frac{1}{24(n+1/2)(t + \log(n+1/2))^2} \\ & \quad + \frac{1}{24n^2(t + \log n)^2} + \frac{1}{12n^2(t + \log n)^3}. \end{aligned}$$

3. One has

$$\theta(t, N) = \theta_n(t, N) + \frac{1 + o(1)}{12n(\log n)^2} \quad (n \rightarrow \infty)$$

and

$$\theta(t, N) = O\left(\frac{1}{Nt^2}\right) \quad (t \rightarrow \infty),$$

where the O constant does not depend on N .

We divide the proof of Theorem 2 into two main steps, based on the equality

$$\theta_n(t, N) = \eta_{n-1}(t, N) + \delta_{n-1}(t, N).$$

First, we prove the following analogue of the theorem for the sequence $\eta_n(t, N)$.

Proposition 2. *Let N be a positive integer, and let $t > -\log N$.*

1. The sequence

$$\eta_n(t, N) = \sum_{k=N}^n \frac{1}{t + \log(k + 1/2)} - \sum_{k=N}^n \frac{1}{H_k - \gamma + t}, \quad n \geq N$$

is positive, strictly increasing, and bounded above. Consequently, the limit

$$\eta(t, N) = \lim_{n \rightarrow \infty} \eta_n(t, N) > 0$$

exists.

2. For all $n \geq N$, one has

$$\frac{1}{24} \int_{n+1}^{\infty} \frac{dx}{x^2(t + \log x)^2} \leq \eta(t, N) - \eta_{n-1}(t, N) = \eta(t, n) \leq \frac{1}{24} \int_n^{\infty} \frac{dx}{x^2(t + \log x)^2}.$$

3. For all $n \geq N$, one has

$$\begin{aligned} & \frac{1}{24(n+1)(t + \log(n+1))^2} - \frac{1}{12(n+1)(t + \log(n+1))^3} \\ & < \eta(t, N) - \eta_{n-1}(t, N) < \frac{1}{24n(t + \log n)^2}. \end{aligned}$$

Consequently, one also has

$$\eta(t, N) = \eta_{n-1}(t, N) + \frac{1 + o(1)}{24n(\log n)^2} \quad (n \rightarrow \infty)$$

and

$$\eta(t, N) = O\left(\frac{1}{Nt^2}\right) \quad (t \rightarrow \infty),$$

where the O constant does not depend on N .

Proof. Let $n \geq N$. By Equation (2.2), for all $k > e^{-t}$, hence for all $k \geq N$, one has

$$0 < \frac{1}{t + \log(k + 1/2)} - \frac{1}{H_k - \gamma + t} = \frac{H_k - \gamma - \log(k + 1/2)}{(t + \log(k + 1/2))(H_k - \gamma + t)} < \frac{1}{24(k + 1/2)^2(t + \log(k + 1/2))^2}.$$

Therefore, the sequence $\eta_n(t, N)$ is positive and strictly increasing, and one has

$$\eta_n(t, N) < \frac{1}{24} \sum_{k=N}^n \frac{1}{(k + 1/2)^2(t + \log(k + 1/2))^2} < \frac{1}{24} \int_N^{n+1} \frac{dx}{x^2(t + \log x)^2}.$$

Similarly, one has

$$\eta_n(t, N) > \frac{1}{24} \sum_{k=N}^n \frac{1}{(k + 1)^2(t + \log(k + 1))^2} > \int_{N+1}^{n+2} \frac{dx}{24x^2(t + \log x)^2},$$

and therefore

$$0 < \frac{1}{24} \int_{N+1}^{n+2} \frac{dx}{x^2(t + \log x)^2} < \eta_n(t, N) < \frac{1}{24} \int_N^{n+1} \frac{dx}{x^2(t + \log x)^2}.$$

Statement (1) follows. Taking the limit as $n \rightarrow \infty$, we see that

$$\frac{1}{24} \int_{N+1}^{\infty} \frac{dx}{x^2(t + \log x)^2} \leq \eta(t, N) \leq \frac{1}{24} \int_N^{\infty} \frac{dx}{x^2(t + \log x)^2}.$$

Statement (2) then follows from the inequality above and the fact that

$$\eta(t, N) - \eta_{n-1}(t, N) = \eta(t, n)$$

for all $n \geq N$. Finally, statement (3) follows from statement (2) and Lemma 1 below. \square

Lemma 1. Let $t, r \in \mathbb{R}$ with $r > 0$ and $t > -\log r$. One has

$$\int_r^{\infty} \frac{dx}{x^2(t + \log x)^2} = e^t \operatorname{li}\left(\frac{1}{re^t}\right) + \frac{1}{r(t + \log r)} > 0.$$

Moreover, for every even positive integer n , one has

$$\sum_{k=2}^{n+1} \frac{(-1)^k (k-1)!}{r(t+\log r)^k} < \int_r^\infty \frac{dx}{x^2(t+\log x)^2} < \sum_{k=2}^n \frac{(-1)^k (k-1)!}{r(t+\log r)^k}.$$

In particular, one has

$$\frac{1}{r(t+\log r)^2} - \frac{2}{r(t+\log r)^3} < \int_r^\infty \frac{dx}{x^2(t+\log x)^2} < \frac{1}{r(t+\log r)^2}.$$

Proof. The exact expression for the integral is easily verified by integration by parts. Since it is known that

$$-\sum_{k=0}^n \frac{(-1)^k k! x}{(\log x)^{k+1}} < \operatorname{li}(x) < -\sum_{k=0}^{n-1} \frac{(-1)^k k! x}{(\log x)^{k+1}}$$

for all $0 < x < 1$ and all even positive integers n , letting $x = \frac{1}{re^t}$, we see that

$$-\sum_{k=1}^n \frac{(-1)^k k!}{r(t+\log r)^{k+1}} < e^t \operatorname{li}\left(\frac{1}{re^t}\right) + \frac{1}{r(t+\log r)} < -\sum_{k=1}^{n-1} \frac{(-1)^k k!}{r(t+\log r)^{k+1}}.$$

The lemma follows. □

The second and final step in the proof of Theorem 2 is to prove the following analogue of the theorem for the sequence $\delta_n(t, N)$.

Proposition 3. *Let N be a positive integer, and let $t > -\log N$.*

1. *The sequence*

$$\delta_n(t, N) = \int_N^{n+1} \frac{dx}{t+\log x} - \sum_{k=N}^n \frac{1}{t+\log(k+1/2)}, \quad n \geq N$$

is positive, strictly increasing, and bounded above. Consequently, the limit

$$\delta(t, N) = \lim_{n \rightarrow \infty} \delta_n(t, N) > 0$$

exists.

2. *For all $n \geq N$, one has*

$$\begin{aligned} & \frac{1}{24(n+1)(t+\log(n+1))^2} \\ & \leq \delta(t, N) - \delta_{n-1}(t, N) = \delta(t, n) \\ & \leq \frac{1}{24(n+1/2)(t+\log(n+1/2))^2} + \frac{1}{24n^2(t+\log n)^2} + \frac{1}{12n^2(t+\log n)^3}. \end{aligned}$$

3. One has

$$\delta(t, N) = \delta_{n-1}(t, N) + \frac{1 + o(1)}{24n(\log n)^2} \quad (n \rightarrow \infty)$$

and

$$\delta(t, N) = O\left(\frac{1}{Nt^2}\right) \quad (t \rightarrow \infty),$$

where the O constant does not depend on N .

Proof. Let $f(x) = \frac{1}{t+\log x}$, which is positive, decreasing, and concave up on $[N, \infty)$. Then, also, $f'(x) = -\frac{1}{x(t+\log x)^2}$ is negative, increasing, and concave down on $[N, \infty)$, while $f''(x) = \frac{1}{x^2(t+\log x)^2} + \frac{2}{x^2(t+\log x)^3}$ is positive, decreasing, and concave up on $[N, \infty)$. In particular, by the well-known expression for the error from the midpoint rule, one has

$$0 < \frac{1}{24}f''(k+1) < \int_k^{k+1} \frac{dx}{t+\log x} - \frac{1}{t+\log(k+1/2)} < \frac{1}{24}f''(k)$$

and therefore

$$\frac{1}{24} \sum_{k=N}^n f''(k+1) < \delta_n(t, N) < \frac{1}{24} \sum_{k=N}^n f''(k).$$

Moreover, one has

$$\sum_{k=N}^n f''(k) < f''(N) + \int_{N+1/2}^{n+1/2} f''(x) dx = f''(N) + f'(n+1/2) - f'(N+1/2),$$

while

$$\sum_{k=N}^n f''(k+1) > \int_{N+1}^{n+2} f''(x) dx = f'(n+2) - f'(N+1).$$

It follows that, for all $n \geq N$, one has

$$\begin{aligned} & \left(\frac{1}{24x(t+\log x)^2}\right)\Big|_{n+2}^{N+1} \\ & < \delta_n(t, N) \\ & < \left(\frac{1}{24x(t+\log x)^2}\right)\Big|_{n+1/2}^{N+1/2} + \frac{1}{24N^2(t+\log N)^2} + \frac{1}{12N^2(t+\log N)^3}. \end{aligned}$$

Statement (1) follows. Statement (2) follows by taking a limit of the inequalities above as $n \rightarrow \infty$, and, finally, statement (3) follows immediately from (2). \square

Theorem 2, now, follows immediately from Propositions 2 and 3. As a corollary, we obtain the following.

Corollary 1. *Let N be a positive integer, and let $t > -\log N$. One has*

$$\theta(t, N) = \eta(t, N) + \delta(t, N).$$

Moreover, for all $n \geq N$, one has

$$0 \leq \theta_n(t, N) < \theta(t, N) < \theta_n(t, N) + \frac{1}{24n(t + \log n)^2} + \frac{1}{24(n + 1/2)(t + \log(n + 1/2))^2} + \frac{1}{24n^2(t + \log n)^2} + \frac{1}{12n^2(t + \log n)^3}.$$

A real function $f(x)$ on an interval $I \subseteq \mathbb{R}$ is said to be *strictly totally monotone* on I if $f(x)$ is continuous on I , infinitely differentiable on the interior of I , and satisfies $(-1)^k \frac{d^k}{dx^k} f(x) > 0$ for all x in the interior of I and for all nonnegative integers k .

Proposition 4. *Let N be a positive integer. The function $\theta(t, N)$ strictly totally monotone on $(-\log N, \infty)$ with*

$$\begin{aligned} (-1)^k \frac{d^k}{dt^k} \theta(t, N) &= (-1)^k \lim_{n \rightarrow \infty} \frac{d^k}{dt^k} \theta_n(t, N) \\ &= k! \lim_{n \rightarrow \infty} \left(\int_N^n \frac{dx}{(t + \log x)^{k+1}} - \sum_{k=N}^{n-1} \frac{1}{(H_k - \gamma + t)^{k+1}} \right) > 0 \end{aligned}$$

for all nonnegative integers k and all $t \in (-\log N, \infty)$.

Proof. Because the sums $\sum_{n=N}^\infty \frac{1}{n(t + \log n)^2}$, $\sum_{n=N}^\infty \frac{1}{n^2(t + \log n)^2}$, and $\sum_{n=N}^\infty \frac{1}{n^2(t + \log n)^3}$ converge for all $t \in (-\log N, \infty)$ and are bounded on compact subsets of $(-\log N, \infty)$, Corollary 1 implies that the convergence of $\theta_n(t, N)$ to $\theta(t, N)$ is uniform on compact subsets of $(-\log N, \infty)$. Therefore, the function $\theta(t, N)$ is continuous on $(-\log N, \infty)$. In fact, the following argument shows that $\theta(t, N)$ is differentiable, with negative derivative, on $(-\log N, \infty)$. First, note that

$$\frac{d}{dt} \theta_n(t, N) = - \int_N^n \frac{dx}{(t + \log x)^2} + \sum_{k=N}^{n-1} \frac{1}{(H_k - \gamma + t)^2},$$

and then a straightforward repetition of our argument for $\theta_n(t, N)$ shows that, just as one has $\theta_n(t, N) > 0$ because

$$\int_N^n \frac{dx}{t + \log x} > \sum_{k=N}^{n-1} \frac{1}{\log(k + 1/2) + t} > \sum_{k=N}^{n-1} \frac{1}{H_k - \gamma + t},$$

one has $\frac{d}{dt} \theta_n(t, N) < 0$ because

$$\int_N^n \frac{dx}{(t + \log x)^2} > \sum_{k=N}^{n-1} \frac{1}{(\log(k + 1/2) + t)^2} > \sum_{k=N}^{n-1} \frac{1}{(H_k - \gamma + t)^2}.$$

One can then bound the error terms as with $\theta(t, N)$ and show that $\frac{d}{dt}\theta_n(t, N)$ converges uniformly on compact subsets of $(-\log N, \infty)$ to a limit

$$f(t, N) = \lim_{n \rightarrow \infty} \frac{d}{dt}\theta_n(t, N)$$

that is negative for all t . It follows that $\theta(t, N)$ is differentiable on $(-\log N, \infty)$ with derivative $f(t, N)$. The same analysis applies more generally to the expression

$$(-1)^k \frac{d^k}{dt^k}\theta_n(t, N) = k! \left(\int_N^n \frac{dx}{(t + \log x)^{k+1}} - \sum_{k=N}^{n-1} \frac{1}{(H_k - \gamma + t)^{k+1}} \right) > 0,$$

for all positive integers k and $n \geq N$. □

Definition 2. Let $t, r \in \mathbb{R}$ with $r > e^{-t}$. Let

$$\beta_x(t, r) = \frac{\text{li}(e^t x)}{e^t} - \sum_{r \leq k < x} \frac{1}{H_k - \gamma + t}$$

for all $x \geq r$, and let

$$\beta(t, r) = \lim_{n \rightarrow \infty} \beta_n(t, r) = \lim_{x \rightarrow \infty} \beta_x(t, r).$$

Let $t, r \in \mathbb{R}$ with $r > e^{-t}$. Since

$$\beta(t, r) = \beta(t, [r]) = \frac{\text{li}(e^t [r])}{e^t} + \theta(t, [r])$$

and

$$\beta(t, r) - \beta_n(t, r) = \beta(t, [r]) - \beta_n(t, [r]) = \theta(t, [r]) - \theta_n(t, [r]) = \theta(t, n)$$

for all $n \geq [r]$, by Theorem 2 one has the following.

Corollary 2. Let $t, r \in \mathbb{R}$ with $r > e^{-t}$. For all $n \geq [r]$, one has

$$\begin{aligned} & \int_{n+1}^{\infty} \frac{dx}{24x^2(t + \log x)^2} + \frac{1}{24(n+1)(t + \log(n+1))^2} \\ & \leq \beta(t, r) - \beta_n(t, r) \\ & \leq \int_n^{\infty} \frac{dx}{24x^2(t + \log x)^2} + \frac{1}{24(n+1/2)(t + \log(n+1/2))^2} \\ & \quad + \frac{1}{24n^2(t + \log n)^2} + \frac{1}{12n^2(t + \log n)^3}. \end{aligned}$$

Moreover, one has

$$\beta(t, r) = \beta_n(t, r) + \frac{1 + o(1)}{12n(\log n)^2} \quad (n \rightarrow \infty).$$

Now, again, let $t, r \in \mathbb{R}$ with $r > e^{-t}$. Then

$$\frac{\text{li}(e^t n)}{e^t} = \sum_{r \leq k < n} \frac{1}{H_k - \gamma + t} + \beta_n(t, r), \quad \forall n \geq r,$$

so that, since by Corollary 2 the quantity $\beta(t, r) - \beta_n(t, r)$ is small for $n \geq r$, one has

$$\frac{\text{li}(e^t n)}{e^t} \approx \sum_{r \leq k < n} \frac{1}{H_k - \gamma + t} + \beta(t, r), \quad \forall n \geq r$$

in a sense made precise by the corollary. It behooves us to choose $r = r_t$ in terms of t so that the absolute value of the quantity

$$\beta(t, r) = \beta(t, \lceil r \rceil) = \frac{\text{li}(e^t \lceil r \rceil)}{e^t} + \theta(t, \lceil r \rceil)$$

is minimized. Since $\theta(t, \lceil r \rceil)$ is always nonnegative, by far the dominant and more unpredictable term in the expression above is $\frac{\text{li}(e^t \lceil r \rceil)}{e^t} = \beta_{\lceil r \rceil}(t, r)$. At least the nonnegativity of $\beta(t, r)$ can be guaranteed as long as $\frac{\text{li}(e^t \lceil r \rceil)}{e^t}$ is nonnegative. So it would be prudent to minimize that term subject to the constraint that $\frac{\text{li}(e^t \lceil r \rceil)}{e^t}$ be nonnegative.

This can be achieved by employing the Ramanujan–Soldner constant μ . Clearly $\frac{\text{li}(e^t \lceil r \rceil)}{e^t}$ is nonnegative if and only if $e^t \lceil r \rceil \geq \mu$, and then the term is minimized for any r with $\lceil r \rceil = \lceil \mu e^{-t} \rceil$, e.g., for $r_t = \mu e^{-t}$, where $r_t > 0$ is uniquely determined by the equation $\text{li}(r_t e^t) = 0$. Let

$$R_t = \lceil r_t \rceil = \lceil \mu e^{-t} \rceil.$$

Since $\text{li}(x)$ is increasing for $x > 1$, one has

$$0 = \frac{\text{li}(e^t r_t)}{e^t} \leq \frac{\text{li}(e^t R_t)}{e^t} < \frac{\text{li}(e^t (r_t + 1))}{e^t} = \frac{\text{li}(\mu + e^t)}{e^t}.$$

Moreover, the function $\frac{\text{li}(\mu + e^t)}{e^t}$ is strictly decreasing with $\lim_{t \rightarrow \infty} \frac{\text{li}(\mu + e^t)}{e^t} = 0$ and

$$\lim_{t \rightarrow -\infty} \frac{\text{li}(\mu + e^t)}{e^t} = \lim_{s \rightarrow 0} \frac{\text{li}(\mu + s) - \text{li}(\mu)}{s} = \text{li}'(\mu) = \frac{1}{\log \mu} = 2.684510350820 \dots$$

Consequently, one has

$$0 \leq \frac{\text{li}(e^t R_t)}{e^t} < \frac{\text{li}(\mu + e^t)}{e^t} < \frac{1}{\log \mu}$$

for all $t \in \mathbb{R}$. Thus, the function $\beta(t, r_t)$ is positive and bounded. More precise bounds on the main term $\frac{\text{li}(e^t R_t)}{e^t}$ of $\beta(t, r_t)$ can be obtained from the following lemma, which follows readily from the fact that the integrand $\frac{1}{\log u}$ of $\text{li}(x) = \int_0^x \frac{du}{\log u}$ is positive, decreasing, and concave up on $(1, \infty)$.

Lemma 2. *One has the following.*

1. *For all $y > x > 1$, one has*

$$0 < \frac{y - x}{\log\left(\frac{x+y}{2}\right)} < \text{li}(y) - \text{li}(x) < \frac{y - x}{\log x}.$$

2. *For all $y > x > 0$ and all $t > -\log x$, one has*

$$0 < \frac{y - x}{t + \log\left(\frac{x+y}{2}\right)} < \frac{\text{li}(e^t y)}{e^t} - \frac{\text{li}(e^t x)}{e^t} < \frac{y - x}{t + \log x}.$$

Corollary 3. *Let $t \in \mathbb{R}$, let $r_t = \mu e^{-t}$, and let $R_t = \lceil r_t \rceil$. One has*

$$0 \leq \beta_{R_t}(t, R_t) = \frac{\text{li}(e^t R_t)}{e^t} < \frac{\text{li}(\mu + e^t)}{e^t} < \frac{1}{\log \mu}$$

and

$$0 \leq \frac{R_t - r_t}{t + \log R_t} \leq \frac{R_t - r_t}{t + \log\left(\frac{R_t + r_t}{2}\right)} \leq \frac{\text{li}(e^t R_t)}{e^t} \leq \frac{R_t - r_t}{\log \mu} < \frac{1}{\log \mu}.$$

The discussion above motivates the following definition.

Definition 3. Let $t \in \mathbb{R}$.

1. Let

$$R_t = \lceil \mu e^{-t} \rceil.$$

Equivalently, R_t is the unique positive integer N such that $t \in [\log \mu - \log N, \log \mu - \log(N - 1))$ (where we set $\log 0 = -\infty$, so that $R_t = 1$ if and only if $t \in [\log \mu, \infty)$).

2. Let

$$\beta_x(t) = \beta_x(t, \mu e^{-t}) = \frac{\text{li}(e^t x)}{e^t} - \sum_{R_t \leq k < x} \frac{1}{H_k - \gamma + t}$$

for all $x \geq \mu e^{-t}$, and let

$$\beta(t) = \beta(t, \mu e^{-t}) = \lim_{n \rightarrow \infty} \beta_n(t) = \lim_{x \rightarrow \infty} \beta_x(t).$$

A graph of the function R_t is provided in Figure 3, and a graph of the function $\beta_{R_t}(t) = \frac{\text{li}(e^t R_t)}{e^t} \approx \beta(t)$ alongside a graph of its upper bound $\frac{\text{li}(\mu + e^t)}{e^t}$ as in Corollary 3 is provided in Figure 4.

Applying Corollary 2 to $r = \mu e^{-t}$, we obtain the following.

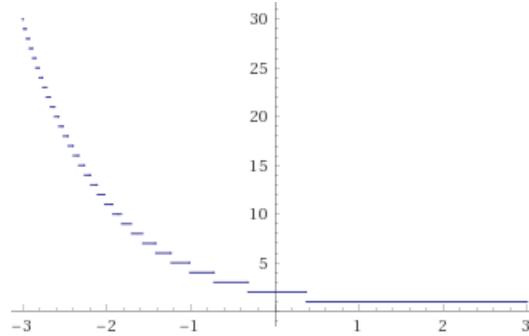


Figure 3: Graph of $R_t = \lceil \mu e^{-t} \rceil$ on $[-3, 3]$

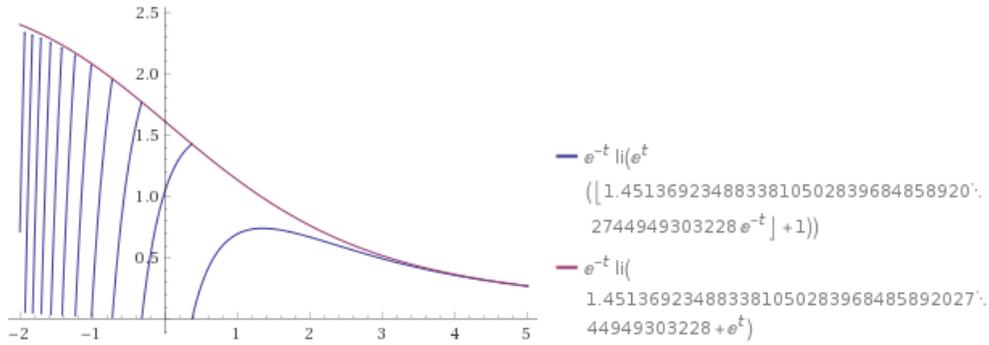


Figure 4: Approximate graph of $\frac{\text{li}(e^t R_t)}{e^t}$ and its upper bound $\frac{\text{li}(\mu + e^t)}{e^t} < \frac{1}{\log \mu}$ on $[-2, 5]$

Corollary 4. For all $t \in \mathbb{R}$ and all $n \geq R_t = \lceil \mu e^{-t} \rceil$, one has

$$\begin{aligned} & \int_{n+1}^{\infty} \frac{dx}{24x^2(t + \log x)^2} + \frac{1}{24(n+1)(t + \log(n+1))^2} \\ & \leq \beta(t) - \beta_n(t) \\ & \leq \int_n^{\infty} \frac{dx}{24x^2(t + \log x)^2} + \frac{1}{24(n + 1/2)(t + \log(n + 1/2))^2} \\ & \quad + \frac{1}{24n^2(t + \log n)^2} + \frac{1}{12n^2(t + \log n)^3}. \end{aligned}$$

Moreover, one has

$$\beta(t) = \beta_n(t) + \frac{1 + o(1)}{12n(\log n)^2} \quad (n \rightarrow \infty).$$

Theorem 1, now, follows immediately from Corollary 4. Letting $n = R_t$ in the corollary, we also conclude the following.

Corollary 5. *For all $t \in \mathbb{R}$, one has*

$$\begin{aligned} & \int_{R_{t+1}}^{\infty} \frac{dx}{24x^2(t + \log x)^2} + \frac{1}{24(R_t + 1)(t + \log(R_t + 1))^2} \\ & \leq \beta(t) - \frac{\text{li}(e^t R_t)}{e^t} \\ & \leq \int_{R_t}^{\infty} \frac{dx}{24x^2(t + \log x)^2} + \frac{1}{24(R_t + 1/2)(t + \log(R_t + 1/2))^2} \\ & \quad + \frac{1}{24R_t^2(t + \log R_t)^2} + \frac{1}{12R_t^2(t + \log R_t)^3}. \end{aligned}$$

Table 1 shows the upper and lower bounds for $\beta(t)$ (rounded up and down, respectively) for all integers $t \in [-15, 15]$ provided by the inequalities in Corollary 5 above and compares them with approximate values of $\beta(t)$ that we computed using WolframAlpha by taking n in the limit expression $\lim_{n \rightarrow \infty} \left(\frac{\text{li}(e^t n)}{e^t} - \sum_{k=\lceil \mu e^{-t} \rceil}^{n-1} \frac{1}{H_k - \gamma + t} \right)$ for $\beta(t)$ as large as the online tool would allow. Notice that the bounds on $\beta(t)$ thus computed for integers $t \in [-15, -7]$ are better than such direct estimates of $\beta(t)$. All of these bounds on $\beta(t)$ can of course be improved by increasing n as in Corollary 4. For instance, in Table 2, we computed the bounds on $\beta(t)$ from Corollary 4 with $n = 50$ for nine special values of t that of particular interest in the next three sections.

Tables 1 and 2 also provide approximate values for the coarsest of all of our lower bounds of $\beta(t)$, namely, $\beta_{R_t}(t) = \frac{\text{li}(e^t R_t)}{e^t}$. In particular, one can see that $\beta(t) \approx \frac{\text{li}(e^t R_t)}{e^t}$ for $|t|$ sufficiently large. This is made precise by the following proposition.

Proposition 5. *Let $t \in \mathbb{R}$.*

1. *One has*

$$\begin{aligned} R_t & \sim \mu e^{-t} \quad (t \rightarrow -\infty), \\ t + \log R_t & \geq \log \mu, \end{aligned}$$

and

$$\limsup_{t \rightarrow -\infty} \frac{\text{li}(e^t R_t)}{e^t} = \lim_{t \rightarrow -\infty} \frac{1}{t + \log R_t} = \text{li}'(\mu) = \frac{1}{\log \mu} = 2.684510350820\dots$$

2. *One has*

$$\beta(t) = \frac{\text{li}(e^t R_t)}{e^t} + \frac{(1 + o(1))e^t}{12\mu(\log \mu)^2} \quad (t \rightarrow -\infty)$$

and

$$\limsup_{t \rightarrow -\infty} \beta(t) = \frac{1}{\log \mu},$$

where also $\frac{1}{12\mu(\log \mu)^2} = -\frac{\text{li}''(\mu)}{12} = \frac{1}{2.416734786403\dots}$.

t	$\beta(t) <$	$\beta(t) \approx$	$\beta(t) >$	$\frac{\text{li}(e^t R_t)}{e^t} \approx$	R_t
15	0.07236490	0.07220	0.07203360	0.07187354	1
14	0.07805640	0.07786	0.07767424	0.07749225	1
13	0.08473348	0.08450	0.08428780	0.08407903	1
12	0.09268293	0.09241	0.09215649	0.09191455	1
11	0.1023178	0.10199	0.1016865	0.1014028	1
10	0.1142551	0.11385	0.1134844	0.1131470	1
9	0.1294542	0.12895	0.1284922	0.1280844	1
8	0.1494680	0.14880	0.1482342	0.1477310	1
7	0.1769053	0.17601	0.1752663	0.1746297	1
6	0.2162592	0.21500	0.2139786	0.2131473	1
5	0.2752827	0.27337	0.2718986	0.2707663	1
4	0.3667085	0.36349	0.3611866	0.3595520	1
3	0.5076564	0.50123	0.4971471	0.4945764	1
2	0.7018947	0.68422	0.6751323	0.6704827	1
1	0.8530561	0.74190	0.7082072	0.6971749	1
0	1.1635319	1.09564	1.0615462	1.0451638	2
-1	0.3044511	0.22299	0.1512070	0.1443674	4
-2	0.7531859	0.74512	0.7346875	0.7160222	11
-3	2.2054409	2.20450	2.2027692	2.1938815	30
-4	2.0135694	2.01342	2.0131210	2.0090540	80
-5	1.6003036	1.60028	1.6002378	1.5986089	216
-6	1.2766878	1.27667	1.2766786	1.2760617	586
-7	1.0210154	1.02099	1.0210141	1.0207849	1592
-8	1.4207283	1.42068	1.4207280	1.4206435	4327
-9	1.1631179	1.16309	1.1631178	1.16308670	11761
-10	1.2488004	1.24879	1.2488003	1.24878890	31969
-11	1.3764747	1.37647	1.3764746	1.37647048	86900
-12	1.8869487	1.88694	1.8869486	1.88694713	236218
-13	2.1844846	2.18448	2.1844845	2.18448397	642106
-14	0.37643374	0.37643	0.37643373	0.37643352	1745423
-15	2.032965028	2.032965	2.032965027	2.032964950	4744552

Table 1: Upper and lower bounds of $\beta(t)$ in Corollary 5 (with $n = R_t$)

t	$\beta(t) <$	$\beta(t) >$	$\frac{\text{li}(e^t R_t)}{e^t}$	R_t
$\gamma + 1 \approx 1.577216$	0.7509547014	0.7509261228	$\text{li}(e^{\gamma+1})/e^{\gamma+1} \approx 0.730170$	1
$\log \alpha \approx 1.347155$	0.7695247294	0.7695229079	$\text{li}(\alpha)/\alpha \approx 0.742305$	1
1	0.7418976158	0.7418955006	$\text{li}(e)/e \approx 0.697175$	1
$\log 2 \approx 0.693147$	0.6026096358	0.6026071971	$\text{li}(2)/2 \approx 0.522582$	1
$\gamma \approx 0.577216$	0.4986013304	0.4985987518	$\text{li}(e^\gamma)/e^\gamma \approx 0.393102$	1
$\log \mu \approx 0.372507$	0.1952555336	0.1952526746	0	1
0	1.0956456993	1.0956421994	$\text{li}(2) \approx 1.045164$	2
$-\log 2 \approx -0.693147$	0.3417372460	0.3417318184	$2 \text{li}(3/2) \approx 0.250130$	3
-1	0.2229882714	0.2229814526	$e \text{li}(4/e) \approx 0.144367$	4

Table 2: Upper and lower bounds of $\beta(t)$ in Corollary 4 with $n = 50$

3. One has

$$\beta(t) < \frac{1}{24}\mu \operatorname{li}\left(\frac{1}{\mu}\right) + \frac{1}{24\log \mu} + \frac{1}{24(\log \mu)^2} + \frac{1}{36(\log(3\mu/2))^2} + \frac{1}{12(\log \mu)^3} = 2.0248039\dots$$

on $[\log \mu, \infty)$.

4. One has

$$\beta(t) = \frac{\operatorname{li}(e^t)}{e^t} + O\left(\frac{1}{t^2}\right) = \frac{1}{t} + O\left(\frac{1}{t^2}\right) \quad (t \rightarrow \infty)$$

and

$$\beta(t) \sim \frac{\operatorname{li}(e^t)}{e^t} \sim \frac{1}{t} \quad (t \rightarrow \infty).$$

5. One has

$$\beta(t) < \limsup_{u \rightarrow -\infty} \beta(u) = \frac{1}{\log \mu},$$

and therefore the least upper bound of the range of β is $\frac{1}{\log \mu}$.

Proof. Statement (1) is an easy consequence of Corollary 3. Moreover, by Corollary 4 and Lemma 1, one has

$$\begin{aligned} & \frac{1}{12(R_t + 1)(t + \log(R_t + 1))^2} - \frac{1}{12(R_t + 1)^2(t + \log(R_t + 1))^3} \\ & < \beta(t) - \frac{\operatorname{li}(e^t R_t)}{e^t} \\ & < \frac{1}{12R_t(t + \log R_t)^2} + \frac{1}{24R_t^2(t + \log R_t)^2} + \frac{1}{12R_t^2(t + \log R_t)^3} \\ & \leq \frac{e^t}{12\mu(\log \mu)^2} + \frac{e^{2t}}{24\mu^2(\log \mu)^2} \left(1 + \frac{2}{\log \mu}\right), \end{aligned} \tag{3.1}$$

and, since

$$\frac{1}{12(R_t + 1)(t + \log(R_t + 1))^2} \sim \frac{1}{12R_t(t + \log R_t)^2} \sim \frac{e^t}{12\mu(\log \mu)^2} \quad (t \rightarrow -\infty),$$

statement (2) follows. By Corollary 4, one has

$$\beta(t) < \frac{\operatorname{li}(e^t)}{e^t} + \frac{1}{24}e^t \operatorname{li}\left(\frac{1}{e^t}\right) + \frac{1}{24t} + \frac{1}{24t^2} + \frac{1}{36(t + \log(3/2))^2} + \frac{1}{12t^3}$$

on $[\log \mu, \infty)$. Moreover, the upper bound of $\beta(t)$ above has negative derivative, and is therefore decreasing, on $[\log \mu, \infty)$. Statement (3) follows. Statement (4) follows Equation (3.1) and the fact that $R_t = 1$ on $[\log \mu, \infty)$.

Finally, we prove statement (5). By (3), we know that $\beta(t) < 2.0248039 \dots < \frac{1}{\log \mu}$ on $[\log \mu, \infty)$. Moreover, by Corollary 3, one has $\frac{\text{li}(e^t R_t)}{e^t} < \frac{\text{li}(\mu + e^t)}{e^t}$. Therefore, by Equation (3.1), one has

$$\beta(t) < \frac{\text{li}(\mu + e^t)}{e^t} + \frac{e^t}{12\mu(\log \mu)^2} + \frac{e^{2t}}{24\mu^2(\log \mu)^2} \left(1 + \frac{2}{\log \mu}\right).$$

Moreover, the upper bound of $\beta(t)$ above is decreasing on $(\infty, -1]$ with limit $\frac{1}{\log \mu}$ as $t \rightarrow -\infty$, and it is less than $\frac{1}{\log \mu}$ also on $[-1, \log(\mu/2)]$. Therefore one has $\beta(t) < \frac{1}{\log \mu}$ on $(-\infty, \log(\mu/2)]$. Moreover, on $[\log(\mu/2), \log \mu]$, one has $R_t = 2$ and therefore, by Corollary 4,

$$\beta(t) < \frac{\text{li}(2e^t)}{e^t} + \frac{1}{24(t + \log 2)^2} + \frac{1}{60(t + \log(5/2))^2} + \frac{1}{96(t + \log 2)^2} + \frac{1}{48(t + \log 2)^3}.$$

Finally, the upper bound of $\beta(t)$ above is maximized on $[\log(\mu/2), \log \mu]$ at the endpoint $\log \mu$, and therefore

$$\begin{aligned} \beta(t) &< \frac{\text{li}(2\mu)}{\mu} + \frac{1}{24(\log(2\mu))^2} + \frac{1}{60(\log(5\mu/2))^2} + \frac{1}{96(\log(2\mu))^2} + \frac{1}{48(\log(2\mu))^3} \\ &= 1.501 \dots < \frac{1}{\log \mu} \end{aligned}$$

on $[\log(\mu/2), \log \mu]$. Thus, we have shown that $\beta(t) < \frac{1}{\log \mu}$ for all $t \in \mathbb{R}$. □

4. Approximating $\pi(x)$ With Harmonic Numbers

The following result of Montgomery and Vaughan [11] is an analogue of Lemma 2 for the prime counting function.

Theorem 3 ([11]). *For all $y > x > 0$ with $y - x > 1$, one has*

$$0 \leq \pi(y) - \pi(x) < \frac{2(y - x)}{\log(y - x)}.$$

The following is an immediate corollary of the result above.

Corollary 6. *Let N be a positive integer. One has the following.*

1. *For all $y > x > 0$ and all $t > -\log(y - x)$, one has*

$$0 \leq \frac{\pi(e^t y)}{e^t} - \frac{\pi(e^t x)}{e^t} < \frac{2(y - x)}{t + \log(y - x)}.$$

2. For all $x \geq 1$ and all $t > -\log N$, one has

$$0 \leq \frac{\pi(e^t x)}{e^t} - \frac{\pi(e^t \lfloor x \rfloor)}{e^t} < \frac{2N}{t + \log N}.$$

The following theorem describes the relevance of the function $\beta(t, r)$ (and thus also the function $\beta(t)$) to the prime counting function.

Theorem 4. *Let δ denote the infimum of the real parts of the zeros of the Riemann zeta function. Suppose that $M > 0$, $C \geq \mu$, and α are constants such that $|\pi(x) - \text{li}(x)| \leq Mx^\alpha \log x$ for all $x \geq C$. Then $\alpha \geq \delta$ and, for all $t, r, \lambda \in \mathbb{R}$ with $\lceil r \rceil \geq \mu e^{-t}$ and $\lambda \geq \beta(t, r)$, and for all integers $n \geq Ce^{-t}$, one has the following.*

1. $\left| \pi(e^t n) - e^t \sum_{r \leq k < n} \frac{1}{H_k - \gamma + t} \right| < Me^{\alpha t} n^\alpha (t + \log n) + \lambda e^t.$
2. $\left| \pi(e^t n) - e^t \sum_{r \leq k < n} \frac{1}{H_k - \gamma + t} \right| < Me^{\alpha t} n^\alpha (H_n - \gamma + t) + \lambda e^t.$
3. $\left| \mathbf{p}(e^t n) - \frac{1}{n} \sum_{r \leq k < n} \frac{1}{H_k - \gamma + t} \right| < \frac{M}{e^{(1-\alpha)t}} \frac{t + \log n}{n^{1-\alpha}} + \frac{\lambda}{n}.$
4. $\left| \mathbf{p}(e^t n) - \frac{1}{n} \sum_{r \leq k < n} \frac{1}{H_k - \gamma + t} \right| < \frac{M}{e^{(1-\alpha)t}} \frac{H_n - \gamma + t}{n^{1-\alpha}} + \frac{\lambda}{n}.$

Conversely, if any of the conditions above hold for all $n \geq C$ for some constants $M > 0$, $C \geq \mu$, $\lambda > 0$, and $t, r, \alpha \in \mathbb{R}$, then $\alpha \geq \delta$, so there exist constants $M' > 0$ and $C' \geq \mu$ such that $|\pi(x) - \text{li}(x)| \leq M'x^\alpha \log x$ for all $x \geq C'$.

Proof. Let α, M, C, t, r , and λ be given as in the forward hypothesis. By Corollary 2, for all $n \geq \lceil r \rceil \geq \mu e^{-t}$, one has

$$0 \leq \frac{\text{li}(e^t n)}{e^t} - \sum_{r \leq k < n} \frac{1}{H_k - \gamma + t} = \beta_n(t, r) < \beta(t, r).$$

Moreover, by hypothesis and the obvious change of variables one has

$$\left| \frac{\pi(e^t n)}{e^t} - \frac{\text{li}(e^t n)}{e^t} \right| \leq \frac{M}{e^{(1-\alpha)t}} n^\alpha (t + \log n)$$

for all $n \geq Ce^{-t} \geq \mu e^{-t}$. Therefore, by the triangle inequality, one has

$$\left| \frac{\pi(e^t n)}{e^t} - \sum_{r \leq k < n} \frac{1}{H_k - \gamma + t} \right| < \frac{M}{e^{(1-\alpha)t}} n^\alpha (t + \log n) + \beta(t, r)$$

for all $n \geq Ce^{-t}$, while also $t + \log n < H_n - \gamma + t$ for all n . The forward direction of the theorem follows.

Conversely, suppose that α, M, C, t, r , and λ are constants satisfying any of the hypotheses (1)–(4). We wish to show that $\alpha \geq \delta$. We may suppose without loss of generality that $\alpha > 0$. Now, since $\log n \sim t + \log n \sim H_n - \gamma + t$ ($n \rightarrow \infty$), any one of statements (1)–(4) implies that

$$\frac{\pi(e^t n)}{e^t} - \sum_{r \leq k < n} \frac{1}{H_k - \gamma + t} = O(n^\alpha \log n) \quad (n \rightarrow \infty),$$

where the O constant depends on the given constants. Moreover, by Corollary 2 one has

$$\frac{\text{li}(e^t n)}{e^t} - \sum_{r \leq k < n} \frac{1}{H_k - \gamma + t} = O(1) \quad (n \rightarrow \infty),$$

so it follows that

$$\frac{\pi(e^t n)}{e^t} - \frac{\text{li}(e^t n)}{e^t} = O(n^\alpha \log n) \quad (n \rightarrow \infty).$$

We wish to show that we can replace the discrete variable n in the above estimate with a continuous variable x . Choose any positive integer $N > e^{-t}$, so that $t > -\log N$. Then, by Corollary 6, for all $x > 1$ one has

$$0 \leq \frac{\pi(e^t x)}{e^t} - \frac{\pi(e^t \lfloor x \rfloor)}{e^t} < \frac{2N}{t + \log N},$$

so that

$$\frac{\pi(e^t x)}{e^t} - \frac{\pi(e^t \lfloor x \rfloor)}{e^t} = O(1) \quad (x \rightarrow \infty).$$

Similarly, by Lemma 2, one has

$$\frac{\text{li}(e^t x)}{e^t} - \frac{\text{li}(e^t \lfloor x \rfloor)}{e^t} = O(1) \quad (x \rightarrow \infty).$$

Therefore, one has

$$\begin{aligned} \frac{\pi(e^t x)}{e^t} - \frac{\text{li}(e^t x)}{e^t} &= \left(\frac{\pi(e^t \lfloor x \rfloor)}{e^t} - \frac{\text{li}(e^t \lfloor x \rfloor)}{e^t} \right) + \left(\frac{\pi(e^t x)}{e^t} - \frac{\pi(e^t \lfloor x \rfloor)}{e^t} \right) - \left(\frac{\text{li}(e^t x)}{e^t} - \frac{\text{li}(e^t \lfloor x \rfloor)}{e^t} \right) \\ &= O(\lfloor x \rfloor^\alpha \log \lfloor x \rfloor) \quad (x \rightarrow \infty) \\ &= O(x^\alpha \log x) \quad (x \rightarrow \infty) \end{aligned}$$

It follows, then, that $\pi(x) - \text{li}(x) = O(x^\alpha \log x)$ ($x \rightarrow \infty$), and therefore $\alpha \geq \delta$. \square

The special case $r = \mu e^{-t}$ yields the following.

Corollary 7. *Let δ denote the infimum of the real parts of the zeros of the Riemann zeta function. Suppose that $M > 0$, $C \geq \mu$, and α are constants such that $|\pi(x) - \text{li}(x)| \leq Mx^\alpha \log x$ for all $x \geq C$. Then $\alpha \geq \delta$ and, for all $t, \lambda \in \mathbb{R}$ with $\lambda \geq \beta(t)$ (e.g., $\lambda = \frac{1}{\log \mu}$), and for all integers $n \geq Ce^{-t}$, one has the following.*

1. $\left| \pi(e^t n) - e^t \sum_{k=\lceil \mu e^{-t} \rceil}^{n-1} \frac{1}{H_k - \gamma + t} \right| < Me^{\alpha t} n^\alpha (t + \log n) + \lambda e^t.$
2. $\left| \pi(e^t n) - e^t \sum_{k=\lceil \mu e^{-t} \rceil}^{n-1} \frac{1}{H_k - \gamma + t} \right| < Me^{\alpha t} n^\alpha (H_n - \gamma + t) + \lambda e^t.$
3. $\left| \mathbf{p}(e^t n) - \frac{1}{n} \sum_{k=\lceil \mu e^{-t} \rceil}^{n-1} \frac{1}{H_k - \gamma + t} \right| < \frac{M}{e^{(1-\alpha)t}} \frac{t + \log n}{n^{1-\alpha}} + \frac{\lambda}{n}.$
4. $\left| \mathbf{p}(e^t n) - \frac{1}{n} \sum_{k=\lceil \mu e^{-t} \rceil}^{n-1} \frac{1}{H_k - \gamma + t} \right| < \frac{M}{e^{(1-\alpha)t}} \frac{H_n - \gamma + t}{n^{1-\alpha}} + \frac{\lambda}{n}.$

Conversely, if any of the conditions above hold for all $n \geq C$ for some constants $M > 0$, $C \geq \mu$, $\lambda > 0$, and $t, r, \alpha \in \mathbb{R}$, then $\alpha \geq \delta$, so there exist constants $M' > 0$ and $C' \geq \mu$ such that

$$|\pi(x) - \text{li}(x)| \leq M' x^\alpha \log x$$

for all $x \geq C'$.

5. Riemann Hypothesis Equivalents Using Harmonic Numbers

In 1976 [14], L. Schoenfeld proved that the Riemann hypothesis is equivalent to Equation (1.2). From Schoenfeld’s result, we can relate Theorem 4 to the Riemann hypothesis as follows.

Theorem 5. *Suppose that the Riemann hypothesis implies that*

$$|\pi(x) - \text{li}(x)| \leq M\sqrt{x} \log x$$

for all $x \geq C$, for constants $M > 0$ and $C \geq \mu$. For example, this implication holds for $M = \frac{1}{8\pi}$ and $C = 2657$. Let $t, r, \lambda \in \mathbb{R}$ with $r > e^{-t}$ and $\lambda \geq \beta(t, r)$. Then the Riemann hypothesis holds if and only if any of the following conditions hold for all $n \geq Ce^{-t}$.

1. $\left| \pi(e^t n) - e^t \sum_{r \leq k < n} \frac{1}{H_k - \gamma + t} \right| \leq M e^{t/2} \sqrt{n} (t + \log n) + \lambda e^t.$
2. $\left| \pi(e^t n) - e^t \sum_{r \leq k < n} \frac{1}{H_k - \gamma + t} \right| \leq M e^{t/2} \sqrt{n} (H_n - \gamma + t) + \lambda e^t.$
3. $\left| \mathbf{p}(e^t n) - \frac{1}{n} \sum_{r \leq k < n} \frac{1}{H_k - \gamma + t} \right| \leq \frac{M}{e^{t/2}} \frac{t + \log n}{\sqrt{n}} + \frac{\lambda}{n}.$
4. $\left| \mathbf{p}(e^t n) - \frac{1}{n} \sum_{r \leq k < n} \frac{1}{H_k - \gamma + t} \right| \leq \frac{M}{e^{t/2}} \frac{H_n - \gamma + t}{\sqrt{n}} + \frac{\lambda}{n}.$

Corollary 8. *Suppose that the Riemann hypothesis implies that*

$$|\pi(x) - \text{li}(x)| \leq M \sqrt{x} \log x$$

for all $x \geq C$, for constants $M > 0$ and $C \geq \mu$. For example, this implication holds for $M = \frac{1}{8\pi}$ and $C = 2657$. Let $t, \lambda \in \mathbb{R}$ with $\lambda \geq \beta(t)$ (e.g., $\lambda = \frac{1}{\log \mu}$). Then the Riemann hypothesis holds if and only if any of the following conditions hold for all $n \geq C e^{-t}$.

1. $\left| \pi(e^t n) - e^t \sum_{r \leq k < n} \frac{1}{H_k - \gamma + t} \right| \leq M e^{t/2} \sqrt{n} (t + \log n) + \lambda e^t.$
2. $\left| \pi(e^t n) - e^t \sum_{r \leq k < n} \frac{1}{H_k - \gamma + t} \right| \leq M e^{t/2} \sqrt{n} (H_n - \gamma + t) + \lambda e^t.$
3. $\left| \mathbf{p}(e^t n) - \frac{1}{n} \sum_{r \leq k < n} \frac{1}{H_k - \gamma + t} \right| \leq \frac{M}{e^{t/2}} \frac{t + \log n}{\sqrt{n}} + \frac{\lambda}{n}.$
4. $\left| \mathbf{p}(e^t n) - \frac{1}{n} \sum_{r \leq k < n} \frac{1}{H_k - \gamma + t} \right| \leq \frac{M}{e^{t/2}} \frac{H_n - \gamma + t}{\sqrt{n}} + \frac{\lambda}{n}.$

Theorem 5 and Corollary 8 can be used to yield a number of arithmetical equivalences of the Riemann hypothesis. For example, letting t equal γ , $\gamma + 1$, 0 , $\log 2$, and 1 , respectively, and employing the upper and lower bounds for $\beta(t)$ provided in Table 2, we obtain the several Riemann hypothesis equivalents listed in Corollaries 9 through 14 below. (For the sake of brevity we express them all in terms of $\mathbf{p}(x)$ instead of $\pi(x)$.)

Corollary 9. *Let $\lambda \geq \beta(\gamma)$, e.g., $\lambda = 0.4986013304$. The Riemann hypothesis holds if and only if any of the following equivalent conditions hold for all integers $n \geq 803$.*

1. $\left| \mathbf{p}(e^\gamma n) - \frac{1}{n} \sum_{k=1}^{n-1} \frac{1}{H_k} \right| < \frac{1}{8\pi e^{\gamma/2}} \frac{\log n + \gamma}{\sqrt{n}} + \frac{\lambda}{n}.$
2. $\left| \mathbf{p}(e^\gamma n) - \frac{1}{n} \sum_{k=1}^{n-1} \frac{1}{H_k} \right| < \frac{1}{8\pi e^{\gamma/2}} \frac{H_n}{\sqrt{n}} + \frac{\lambda}{n}.$

Proof. By Corollary 8, the Riemann hypothesis holds if and only if any of the conditions hold for all $n \geq 2657e^{-\gamma} > 1491$, and the given inequalities can be verified directly to hold for all $803 \leq n \leq 1491$, even for the lower bound $\lambda = 0.49859$ of $\beta(\gamma)$. □

Corollary 10. *Let $\lambda \geq \beta(\gamma) + 1 = \beta(\gamma, \mu e^{-\gamma} + 1)$, e.g. $\lambda = 1.4986013304$. The Riemann hypothesis holds if and only if any of the following equivalent conditions hold for all positive integers n .*

1. $\left| \mathbf{p}(e^\gamma n) - \frac{1}{n} \sum_{k=2}^{n-1} \frac{1}{H_k} \right| < \frac{1}{8\pi e^{\gamma/2}} \frac{\log n + \gamma}{\sqrt{n}} + \frac{\lambda}{n}.$
2. $\left| \mathbf{p}(e^\gamma n) - \frac{1}{n} \sum_{k=2}^{n-1} \frac{1}{H_k} \right| < \frac{1}{8\pi e^{\gamma/2}} \frac{H_n}{\sqrt{n}} + \frac{\lambda}{n}.$

Proof. By Theorem 5, the Riemann hypothesis holds if and only if any of the conditions hold for all $n \geq 2657e^{-\gamma} > 1491$, and the given inequalities can be verified directly to hold for all $1 \leq n \leq 1491$, even for the lower bound $\lambda = 1.49859$ of $\beta(\gamma) + 1$. □

Corollary 11. *Let $\lambda \geq \beta(\gamma + 1)$, e.g. $\lambda = 0.7509547014$. The Riemann hypothesis holds if and only if any of the following equivalent conditions hold for all positive integers n .*

1. $\left| \mathbf{p}(e^{\gamma+1} n) - \frac{1}{n} \sum_{k=1}^{n-1} \frac{1}{H_k + 1} \right| < \frac{1}{8\pi e^{(\gamma+1)/2}} \frac{\log n + \gamma + 1}{\sqrt{n}} + \frac{\lambda}{n}.$
2. $\left| \mathbf{p}(e^{\gamma+1} n) - \frac{1}{n} \sum_{k=1}^{n-1} \frac{1}{H_k + 1} \right| < \frac{1}{8\pi e^{(\gamma+1)/2}} \frac{H_n + 1}{\sqrt{n}} + \frac{\lambda}{n}.$

Proof. By Corollary 8, the Riemann hypothesis holds if and only if any of the conditions hold for all $n \geq 2657e^{-\gamma-1} > 548$, and the given inequalities can be verified directly to hold for all $1 \leq n \leq 548$, even for the lower bound $\lambda = 0.750926$ of $\beta(\gamma)$. □

Corollary 12. *Let $\lambda \geq \beta(0)$, e.g., $\lambda = 1.0956456993$. The Riemann hypothesis holds if and only if any of the following equivalent conditions hold for all positive integers $n \geq 1427$.*

1. $\left| \mathbf{p}(n) - \frac{1}{n} \sum_{k=2}^{n-1} \frac{1}{H_k - \gamma} \right| < \frac{1}{8\pi} \frac{\log n}{\sqrt{n}} + \frac{\lambda}{n}.$
2. $\left| \mathbf{p}(n) - \frac{1}{n} \sum_{k=2}^{n-1} \frac{1}{H_k - \gamma} \right| < \frac{1}{8\pi} \frac{H_n - \gamma}{\sqrt{n}} + \frac{\lambda}{n}.$

Proof. By Corollary 8, the Riemann hypothesis holds if and only if any of the conditions hold for all $n \geq 2657$, and the given inequalities can be verified directly to hold for all $1427 \leq n \leq 2657$, even for the lower bound $\lambda = 1.09564$ of $\beta(\gamma)$. \square

Corollary 13. *Let $\lambda \geq \beta(\log 2)$, e.g., $\lambda = 0.6026096358$. The Riemann hypothesis holds if and only if any of the following equivalent conditions hold for all positive integers $n \geq 714$.*

1. $\left| \mathbf{p}(2n) - \frac{1}{n} \sum_{k=1}^{n-1} \frac{1}{H_k - \gamma + \log 2} \right| < \frac{1}{8\pi} \frac{\log(2n)}{\sqrt{2n}} + \frac{\lambda}{n}.$
2. $\left| \mathbf{p}(2n) - \frac{1}{n} \sum_{k=1}^{n-1} \frac{1}{H_k - \gamma + \log 2} \right| < \frac{1}{8\pi} \frac{H_n - \gamma + \log 2}{\sqrt{2n}} + \frac{\lambda}{n}.$

Proof. By Corollary 8, the Riemann hypothesis holds if and only if any of the conditions hold for all $n \geq 2657/2 > 1328$, and the given inequalities can be verified directly to hold for all $714 \leq n \leq 1328$, even for the lower bound $\lambda = 0.602607$ of $\beta(\gamma)$. \square

Corollary 14. *Let $\lambda \geq \beta(1)$, e.g., $\lambda = 0.7418976158$. The Riemann hypothesis holds if and only if any of the following equivalent conditions hold for all positive integers $n \neq 82$.*

1. $\left| \mathbf{p}(en) - \frac{1}{n} \sum_{k=1}^{n-1} \frac{1}{H_k - \gamma + 1} \right| < \frac{1}{8\pi} \frac{\log(en)}{\sqrt{en}} + \frac{\lambda}{n}.$
2. $\left| \mathbf{p}(en) - \frac{1}{n} \sum_{k=1}^{n-1} \frac{1}{H_k - \gamma + 1} \right| < \frac{1}{8\pi} \frac{H_n - \gamma + 1}{\sqrt{en}} + \frac{\lambda}{n}.$

Proof. By Corollary 8, the Riemann hypothesis holds if and only if any of the conditions hold for all $n \geq 2657e^{-1} > 977$, and the given inequalities can be verified directly to hold for all $1 \leq n \leq 977$ with $n \neq 82$, even for the lower bound $\lambda = 0.741895$ of $\beta(\gamma)$. \square

Curiously, $n = 82$ is the only (hypothetical) exception to the two Riemann hypothesis equivalents in Corollary 14.

Our results specialize to O bounds as follows.

Theorem 6. *Let δ be the supremum of the real parts of the zeros of the Riemann zeta function, let N be a positive integer, and let $t \in \mathbb{R}$ so that $t \neq \gamma - H_n$ for all $n \geq N$ (which holds if $t > \gamma - H_N$). Then δ is the smallest real number α such that*

$$\pi(e^t n) = e^t \sum_{k=N}^{n-1} \frac{1}{H_k - \gamma + t} + O(n^\alpha H_n) \quad (n \rightarrow \infty).$$

In particular, the Riemann hypothesis is equivalent to

$$\pi(e^t n) = e^t \sum_{k=N}^{n-1} \frac{1}{H_k - \gamma + t} + O(\sqrt{n} H_n) \quad (n \rightarrow \infty).$$

Corollary 15. *The Riemann hypothesis is equivalent to*

$$\mathbf{p}(e^\gamma n) = \frac{1}{n} \sum_{k=1}^{n-1} \frac{1}{H_k} + O\left(\frac{H_n}{\sqrt{n}}\right) \quad (n \rightarrow \infty)$$

and to

$$\mathbf{p}(n) = \frac{1}{n} \sum_{k=1}^{n-1} \frac{1}{H_k - \gamma} + O\left(\frac{H_n}{\sqrt{n}}\right) \quad (n \rightarrow \infty).$$

For any positive real numbers x_1, x_2, \dots, x_n , let

$$M_{-1}(x_1, x_2, \dots, x_n) = \frac{n}{x_1^{-1} + x_2^{-1} + \dots + x_n^{-1}}$$

denote the harmonic mean of x_1, x_2, \dots, x_n . Thus, for example, one has

$$M_{-1}(1, 2, 3, \dots, n) = \frac{n}{H_n}$$

for all positive integers n . Since $\frac{1}{H_n - \gamma + t} = o(1)$ ($n \rightarrow \infty$), one can replace the upper limits $x - 1$ and $n - 1$ of the sums in Theorem 6 and Corollary 15 with x and n , respectively. Thus, we have the following.

Corollary 16. *Let $t > -1$. Each of the following statements is equivalent to the Riemann hypothesis.*

1. $\mathbf{p}(e^{t+\gamma} n) = \frac{1}{M_{-1}(H_1 + t, H_2 + t, \dots, H_n + t)} + O\left(\frac{H_n}{\sqrt{n}}\right) \quad (n \rightarrow \infty).$
2. $\frac{1}{\mathbf{p}(e^{t+\gamma} n)} = M_{-1}(H_1 + t, H_2 + t, \dots, H_n + t) + O\left(\frac{H_n^3}{\sqrt{n}}\right) \quad (n \rightarrow \infty).$

3. $\mathbf{p}(e^\gamma n) = \frac{1}{M_{-1}(H_1, H_2, \dots, H_n)} + O\left(\frac{H_n}{\sqrt{n}}\right) \quad (n \rightarrow \infty).$
4. $\frac{1}{\mathbf{p}(e^\gamma n)} = M_{-1}(H_1, H_2, \dots, H_n) + O\left(\frac{H_n^3}{\sqrt{n}}\right) \quad (n \rightarrow \infty).$
5. $\mathbf{p}(n) = \frac{1}{M_{-1}(H_1 - \gamma, H_2 - \gamma, \dots, H_n - \gamma)} + O\left(\frac{H_n}{\sqrt{n}}\right) \quad (n \rightarrow \infty).$
6. $\frac{1}{\mathbf{p}(n)} = M_{-1}(H_1 - \gamma, H_2 - \gamma, \dots, H_n - \gamma) + O\left(\frac{H_n^3}{\sqrt{n}}\right) \quad (n \rightarrow \infty).$

6. Monotonicity Properties of the Error Term $\beta(t)$

In this final section, we examine the intervals of increase and decrease of the function

$$\beta(t) = \lim_{n \rightarrow \infty} \left(\frac{\text{li}(e^t n)}{e^t} - \sum_{k=\lceil \mu e^{-t} \rceil}^{n-1} \frac{1}{H_k - \gamma + t} \right).$$

By Proposition 4, the function $\theta(t, 1)$ is strictly totally monotone on the interval $(0, \infty)$. Since $\theta(t, 1) = \beta(t) - \frac{\text{li}(e^t)}{e^t}$ on $[\log \mu, \infty)$, it follows that $\beta(t) - \frac{\text{li}(e^t)}{e^t}$ is strictly totally monotone on the interval $[\log \mu, \infty)$. Let $\alpha = 3.846467717046\dots$ denote the unique zero of $\frac{d}{dx} \frac{\text{li}(x)}{x}$, which is also the unique solution to the equation $\text{li}(x) = \frac{x}{\log x}$. Alternatively, $\log \alpha = 1.347155251069\dots$ is the unique zero of $\frac{d}{dx} \frac{\text{li}(e^x)}{e^x}$ and is the unique solution to the equation $x \text{li}(e^x) = e^x$. The function $\frac{\text{li}(x)}{x}$ is strictly increasing on $(1, \alpha]$ and strictly decreasing on $[\alpha, \infty)$. Likewise, the function $\frac{\text{li}(e^x)}{e^x}$ is strictly increasing on $(0, \log \alpha]$ and strictly decreasing on $[\log \alpha, \infty)$.

Proposition 6. *Let N be a positive integer. One has the following.*

1. *The function $\beta(t) - \frac{\text{li}(e^t R_t)}{e^t}$ is strictly totally monotone on the interval $[\log \mu, \infty)$ (where $R_t = 1$). Moreover, the function $\beta(t)$ is strictly decreasing on the interval $[\log \alpha, \infty)$, and therefore $\beta(t) \leq \beta(\log \alpha) < 0.7695247294$ on $[\log \alpha, \infty)$.*
2. *The function $\beta(t) - \frac{\text{li}(e^t R_t)}{e^t}$ is strictly totally monotone; the function $\frac{\text{li}(e^t R_t)}{e^t}$ is strictly increasing and concave down, on the interval $[\log(\mu/N), \log(\mu/(N - 1))]$, if $N \geq 2$.*
3. *$\beta(t)$ is strictly increasing on the interval $[\log(\mu/N), \log(\mu/(N - 1))]$ if $N \geq 3$.*
4. *$\beta(t)$ is strictly increasing on the interval $[\log(\mu/2), \log(\alpha/3)]$ (on which $R_t = 2$).*

Proof. We have already proved statement (1), so we may suppose that $R_t \geq 2$. For $t \in I = [\log(\mu/N), \log(\mu/(N - 1))]$, the function $R_t = N$ is constant. Therefore, by Proposition 4, the function $\beta(t) - \frac{\text{li}(e^t R_t)}{e^t} = \theta(t, N)$ is strictly totally monotone on I . Moreover, since $e^t N \leq \mu N / (N - 1) \leq 2\mu < \alpha$ on I , the function $\frac{\text{li}(e^t R_t)}{e^t} = \frac{\text{li}(e^t N)}{e^t}$ is strictly increasing on I . Furthermore, the derivative $\frac{d}{dt} \frac{\text{li}(e^t N)}{e^t} = \frac{N}{t + \log N} - \frac{\text{li}(e^t N)}{e^t}$ is decreasing on I , so that $\frac{\text{li}(e^t R_t)}{e^t}$ is concave down on I . Again by Proposition 4, one has

$$\begin{aligned} \beta'(t) &= \lim_{n \rightarrow \infty} \left(\frac{d}{dt} \frac{\text{li}(e^t n)}{e^t} + \sum_{k=N}^{n-1} \frac{1}{(H_k - \gamma + t)^2} \right) \\ &\geq \lim_{n \rightarrow \infty} \left(\frac{d}{dt} \frac{\text{li}(e^t n)}{e^t} + \sum_{k=N}^{n-1} \frac{1}{(t + \log(k + 1))^2} \right) \\ &\geq \lim_{n \rightarrow \infty} \left(\frac{d}{dt} \frac{\text{li}(e^t n)}{e^t} + \int_{N+1}^{n+1} \frac{dx}{(t + \log x)^2} \right) \\ &= \lim_{n \rightarrow \infty} \left(- \int_{\alpha e^{-t}}^n \frac{dx}{(t + \log x)^2} + \int_{N+1}^{n+1} \frac{dx}{(t + \log x)^2} \right) \\ &= \lim_{n \rightarrow \infty} \left(\int_{N+1}^{\alpha e^{-t}} \frac{dx}{(t + \log x)^2} + \int_n^{n+1} \frac{dx}{(t + \log x)^2} \right) \\ &= \int_{R_t+1}^{\alpha e^{-t}} \frac{dx}{(t + \log x)^2}. \end{aligned}$$

Moreover, one has $\alpha e^{-t} > R_t + 1$ provided that $R_t \geq 3$ since $t < \log(\mu/(R_t - 1)) < \log(\alpha/(R_t + 1))$ if $R_t \geq 3$. Therefore $\beta'(t) \geq \int_{N+1}^{\alpha e^{-t}} \frac{dx}{(t + \log x)^2} > 0$ if $N \geq 3$. Finally, if $R_t = 2$, then $\alpha e^{-t} > R_t + 1 = 3$ provided that $t < \log(\alpha/3)$, so that $\beta'(t) > 0$ on $[\log(\mu/2), \log(\alpha/3)]$. \square

Thus, $\beta(t)$ is strictly increasing on $[\log(\mu/N), \log(\mu/(N - 1))]$ as long as $N \geq 3$, but the cases $N = 1$ and $N = 2$ are still somewhat of a mystery, since we only know that $\beta(t)$ is strictly decreasing on $[\log \alpha, \infty) \subsetneq [\log \mu, \infty)$ and $\beta(t)$ is strictly increasing on $[\log(\mu/2), \log(\alpha/3)] \subsetneq [\log(\mu/2), \log \mu)$. The only remaining intervals to examine, then, are $[\log \mu, \log \alpha]$ and $[\log(\alpha/3), \log \mu)$.

Let us examine the first interval. Since $\frac{\text{li}(e^x)}{e^x}$ is a reasonable approximation (and lower bound) of $\beta(t)$ on $[\log \mu, \infty)$, one might expect that there exists a constant $c \geq \log \mu$ such that $\beta(t)$ is increasing on $[\log \mu, c]$ and decreasing on $[c, \infty)$. This expectation is realized if the following two plausible conjectures hold: (1) for all positive integers n , function $\beta_n(t) = \frac{\text{li}(ne^t)}{e^t} - \sum_{k=1}^{n-1} \frac{1}{H_k - \gamma + t}$ has a unique local maximum on $[\log \mu, \infty)$ at some $\rho_n \in (\log \mu, \log \alpha)$, and (2) the limit $\rho = \lim_{n \rightarrow \infty} \rho_n$ exists. (Numerical evidence leads one to suspect further that: (3) the ρ_n are bounded below by ρ , and (4) the ρ_n are strictly decreasing as $n \rightarrow \infty$, which, together with

n	ρ_n
1	$\log \alpha \approx 1.347155$
2	≈ 1.29475
3	≈ 1.28724
4	≈ 1.28489
5	≈ 1.28386
6	≈ 1.28331
7	≈ 1.28298
8	≈ 1.28277
9	≈ 1.2826260
10	≈ 1.2825221
4000	≈ 1.282020
5000	≈ 1.282020

Table 3: Local maximum of $\beta_n(t)$ on $[\log \mu, \infty)$ attained at $t = \rho_n$

(1), would imply (2).) Suppose, for the sake of argument, that conjectures (1) and (2) are true. Let $\varepsilon > 0$. Then the $\beta_n(t)$ are decreasing on $[\rho + \varepsilon, \infty)$ for sufficiently large n , whence $\beta(t) = \lim_{n \rightarrow \infty} \beta_n(t)$ is also decreasing on $[\rho + \varepsilon, \infty)$. At the same time, the $\beta_n(t)$ are increasing on $[\log \mu, \rho - \varepsilon]$ for sufficiently large n , so that $\beta(t)$ is increasing on $[\log \mu, \rho - \varepsilon]$. Therefore, if conjectures (1) and (2) are true, then $\beta(t)$ is increasing on $[\log \mu, \rho]$ and decreasing on $[\rho, \infty)$, and therefore $\beta(t)$ attains a local maximum at $t = \rho$. Table 3 lists approximate values of ρ_n for $n = 1, 2, 3, \dots, 10$, where $\beta_n(t)$ attains a unique local maximum at the given values of $t = \rho_n$, and also for $n = 4000$ and $n = 5000$, where $\beta_n(t)$ attains at least one local maximum at $t \approx 1.28202$. Thus, from the computations in Table 3, it appears that $\rho \approx 1.28202$ exists. A separate calculation, shown in Table 4, shows that indeed $\beta(t)$ attains at least one local maximum value of approximately 0.77067 at some t near 1.282. More precisely, from the calculations in Table 4 one has

$$\beta(1.274) < 0.770653 < 0.770657 < \beta(1.282)$$

and

$$\beta(1.290) < 0.770653 < 0.770657 < \beta(1.282),$$

and therefore, since $\beta(t)$ is differentiable on $(\log \mu, \infty)$, one has the following.

Proposition 7. *The function $\beta(t)$ attains at least one local maximum value at some $t = \rho$ satisfying $1.274 < \rho < 1.290$.*

A similar analysis of $\beta(t)$ on the interval $[\log(\mu/2), \log \mu)$ suggests that $\beta(t)$ is strictly increasing on the entire interval, not just on $[\log(\mu/2), \log(\alpha/3)]$. Thus we pose the following.

Conjecture 1. *There exists a constant $\rho \in (\log \mu, \log \alpha)$ ($\rho \approx 1.282$) such that $\beta(t)$ is strictly increasing on $[\log \mu, \rho]$ and strictly decreasing on $[\rho, \infty)$. Moreover, $\beta(t)$ is strictly increasing on $[\log(\mu/2), \log \mu)$.*

t	$\beta(t) <$	$\beta(t) \approx$	$\beta(t) >$
1.274	0.770653	0.770651	0.770639
1.280	0.770670	0.770668	0.770656
1.281	0.770671	0.770669	0.770657
1.282	0.770671	0.770669	0.770657
1.283	0.770671	0.770669	0.770657
1.284	0.770670	0.770668	0.770656
1.285	0.770669	0.770667	0.770655
1.290	0.770653	0.770651	0.770663

Table 4: Upper and lower bounds of $\beta(t)$ computed with $n = 100$, and approximations of $\beta(t)$ with $n = 1000$

References

- [1] H. Alzer, On some inequalities for the gamma and psi functions, *Math. Comput.* **66**, (1997), no. 217, 373–389.
- [2] C.-J. de la Vallée Poussin, Recherches analytiques la théorie des nombres premiers, *Ann. Soc. scient. Bruxelles* **20**, (1896), 183–256.
- [3] C.-J. de la Vallée Poussin, Sur la fonction Zeta de Riemann et le nombre des nombres premiers inferieur a une limite donnée, *C. Mém. Couronnés Acad. Roy. Belgique* **59**, (1899), 1–74.
- [4] J. Elliott, Asymptotic expansions of the prime counting function, arXiv:1809.06633v4 [math.NT], submitted.
- [5] K. Ford, Vinogradov’s integral and bounds for the Riemann zeta function, *Proc. London Math. Soc.* **85**, (2002), no. 3, 565–633.
- [6] T. H. Gronwall, Some asymptotic expressions in the theory of numbers, *Trans. Amer. Math. Soc.* **14**, (1913), 113–122.
- [7] J. Hadamard, Sur la distribution des zéros de la fonction $\zeta(s)$ et ses conséquences arithmétiques, *Bull. Soc. math. France* **24**, (1896), 199–220.
- [8] H. von Koch, Sur la distribution des nombres premiers, *Acta Mathematica* **24**, (1901), 159–182.
- [9] J. C. Lagarias, An elementary problem equivalent to the Riemann hypothesis, *Amer. Math. Monthly*, **109**, (2002), no. 6, 534–543.
- [10] F. Mertens, Ein Beitrag zur analytischen Zahlentheorie, *J. reine angew. Math.* **78**, (1874), 46–62.
- [11] H. L. Montgomery and R. C. Vaughan, The large sieve, *Mathematika* **20**, (1973), no. 2, 119–134.
- [12] H. L. Montgomery and R. C. Vaughan, *Multiplicative Number Theory I. Classical Theory*, Cambridge Studies in Advanced Mathematics **97**, Cambridge University Press, 2007.
- [13] G. Robin, Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann, *J. Math. Pures Appl.* **63**, (1984), 187–213.
- [14] L. Schoenfeld, Sharper bounds for the Chebyshev functions $\theta(x)$ and $\psi(x)$. II., *Math. Comput.* **30**, (1976), no. 134, 337–360.