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ON THE TOTAL NUMBER OF PRIME FACTORS OF AN ODD PERFECT NUMBER

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Abstract

Let N be an odd perfect number. Let $\omega(N)$ be the number of distinct prime factors of N and let $\Omega(N)$ be the total number of prime factors of N. We prove that if $(3, N) = 1$, then $\frac{302}{113}\omega(N) - \frac{286}{113} \leq \Omega(N)$. If $3 \mid N$, then $\frac{66}{25}\omega(N) - 5 \leq \Omega(N)$. This is an improvement on similar prior results by the author which was an improvement of a result of Ochem and Rao. We also establish new lower bounds on $\omega(N)$ in terms of the smallest prime factor of N and establish new lower bounds on N in terms of its smallest prime factor.

1. Introduction

Recall that a positive integer N is said to be perfect if the sum of N's proper divisors add up to n, or equivalently, that $\sigma(N) = 2N$, where $\sigma(N)$ is the sum of the divisors of N.

It is currently unknown whether there are any odd perfect numbers. Let N be an odd perfect number. Ochem and Rao [16] have proved that N must satisfy

$$\Omega(N) \geq \frac{18\omega(N) - 31}{7} \tag{1}$$

and

$$\Omega(N) \geq 2\omega(n) + 51. \tag{2}$$

Here $\Omega(N)$ is the total number of prime divisors of N, and $\omega(N)$ is the number of distinct prime divisors of N. Note that Ochem and Rao's second inequality is stronger than the first as long as $\omega(N) \leq 81$. Nielsen [13] has shown that $\omega(n) \geq 10$. In a previous paper [23], the author improved on Ochem and Rao's result. In that paper the following theorem was proved.

[1]This paper was primarily written while the author was a lecturer at Iowa State University.

Theorem 1. *If N is an odd perfect number with $3 \not| N$, then*

$$\Omega(N) \geq \frac{8}{3}\omega(N) - \frac{7}{3}. \tag{3}$$

If N is an odd perfect number with $3|N$, then

$$\Omega(N) \geq \frac{21}{8}\omega(N) - \frac{39}{8}. \tag{4}$$

.

In this paper we prove improve on that result as follows.

Theorem 2. *If $3 \nmid N$, then*

$$\Omega(N) \geq \frac{302}{113}\omega(N) - \frac{286}{113}. \tag{5}$$

If $3 \mid N$, then

$$\Omega(N) \geq \frac{66}{25}\omega(N) - 5. \tag{6}$$

Note that while Inequality (6) is always better than Inequality (4), Inequality (5) is only better than Inequality (3) when $\omega \geq 34$. Note that the worst case of the above is when $3 \mid N$, and so we have the following corollary.

Corollary 1. *If N is an odd perfect number then*

$$\Omega(N) \geq \frac{66}{25}\omega(N) - 5.$$

Note that Kevin Hare [9] has shown that in general any odd perfect number must satisfy $\Omega(N) \geq 75$, while [15] has improved this to $\Omega(N) \geq 101$.

This paper contains eight sections. The first section is the introduction. The second section contains various results we will need to prove Theorem 2. The third section contains the proof of Theorem 2 when $3 \mid N$. The fourth section contains the proof when $(3, N) = 1$. The fifth section improves on the known lower bound of $\omega(N)$ in terms of the smallest prime factor of N. This is essentially a small improvement of existing results although new questions are raised based on some aspects of the methods used. The sixth section combines the ideas of the previous sections to improve lower bounds for N in terms of its smallest prime factor. The seventh section discusses a new way of measuring the strength of a statement about odd perfect numbers and evaluates the Ochem and Rao type bounds in that context. The eighth section discusses various related open problems that are naturally connected to improving these results. We will use the following notation: N will be an odd perfect number. We will write Ω for $\Omega(N)$ and write ω for $\omega(N)$. We recall Euler's classical theorem on odd perfect numbers. Euler proved that N must have the form $N = q^e m^2$ where q is a prime such that $q \equiv e \equiv 1 \pmod 4$ and $(q, m) = 1$. Traditionally, q is called the special prime.[2] Note that from Euler's result one immediately has $\Omega \geq 2\omega - 1$. Essentially all improvements on Ochem-Rao type inequalities can be thought of as improving on the bound from Euler's theorem. For the remainder of this paper we will assume that N is an odd perfect number with q, e and m given as above. The basic method of this paper is the same as that used in Ochem and Rao's result. The essential observation is that if N is an odd perfect number with a prime p raised to just the second power, then for each such prime p we have $(p^2 + p + 1)|n$. It follows from quadratic reciprocity that if q is a prime and $q|(p^2+p+1)$, then q is either equal to 3 or is 1 mod 3. Now assume that N has many such primes p. If $p^2 + p + 1$ is often divisible by 3, then N will be divisible by a large power of 3. If q is not 3 in some instance, then with the exception of q being the special prime, one has one of two situations: Either one has $q^2||N$, which gives a new 3 in the factorization since $q \equiv 1 \pmod 3$, and so $3|(q^2 + q + 1)$, or one at least has $q^4|N$. The key is that if one has many numbers of the form $p^2 + p + 1$ then there is not much room for a lot of primes to be repeated exactly twice. And if one has most primes raised to a higher power then one gets repeated prime factors from those primes. This idea is made rigorous through a system of linear inequalities. Finding the optimum of the resulting linear program gives the desired inequality.

While this paper is substantially longer than the previous paper by this author and Ochem and Rao's paper, the basic method remains the same. The improvements in the case when $3 \mid N$ are essentially straightforward and represent a small improvement of the technique using some new minor number theoretic results to get additional inequalities in the system used. The case when $(3, N) = 1$ involves three major new ingredients. Our first new ingredient is the notion of a triple threat. Define a *triple threat* to be a quadruple of four odd primes x, a, b, and c such that

$$\sigma(x^2) = x^2 + x + 1 = \sigma(a^2)(\sigma(b^2)\sigma(c^2) = (a^2 + a + 1)(b^2 + b + 1)(c^2 + c + 1)$$

and where $a^2 + a + 1$, $b^2 + b + 1$, and $c^2 + c + 1$ are all prime. The primary obstruction to improving the $\frac{8}{3}$ bound in Theorem 1 arose from the fact that we could not rule out the existence of odd perfect numbers with many prime divisors forming triple threats. Consider a triple threat (x, a, b, c) where $x \equiv a \equiv 1 \pmod 5$. One of our results in this paper is that no such triple threats exist with this property. This restriction of triple threats is not by itself sufficient to improve the bound. Our second new ingredient is that if $p^4||N$, then every prime divisor of $\sigma(p^4)$ is either equal to 5 or is congruent to 1 modulo 5. Our third new ingredient is the

[2]Some authors call q the "Euler prime." A better name in fact would be the Cartesian prime, since prior to Euler's result Descartes proved that an odd perfect number needed to have exactly one prime factor raised to an odd power. In any event, the term special prime avoids any issues of priority.

observation that we have the following surprising factorization:[3] If $f(x) = x^2+x+1$, and $g(x) = x^4 + x^3 + x^2 + x + 1$, then

$$f(g(x)) = (x^2 - x + 1)(x^6 + 3x^5 + 5x^4 + 6x^3 + 7x^2 + 6x + 3).$$

This means that if we have a prime m such that $m^4||N$, and $\sigma(m^4) = p$ is itself prime, then we can substantially restrict what $\sigma(p^2)$ looks like. We can combine the second and third ingredients to guarantee that one of the following holds: (a) there are many primes p where $p^2||N$ and $p \equiv 1 \pmod 5$; (b) N is divisible by a large power of 5; (c) N has many prime factors raised to at least the sixth power; (d) N has many prime factors of a very special form arising from the third ingredient factorization. Note that all three of these ingredients are necessary for our improvement. Any two of them will not by themselves give rise to an improvement beyond the $\frac{8}{3}$ bound.

2. Foundations

This section contains various lemmata we will need for the main results. We will assume some familiarity with the literature on perfect numbers but will recall some basic facts here. For some early history on this matter see [5].

While the ancient Greeks originally defined perfect numbers in terms of the sum of the proper divisors, it is more natural for most purposes to define a number n as perfect if n satisfies $\sigma(n) = 2n$ where $\sigma(n)$ is the sum of all the positive divisors of n. Much of the study of perfect numbers relies on the nice fact that $\sigma(n)$ is a multiplicative function. Recall that a number n is said to be *abundant* if $\sigma(n) > 2n$, and it is is said to be *deficient* if $\sigma(2n) < n$. We will set $h(n) = \frac{\sigma(n)}{n}$. Note that $h(n)$ has three names in the literature. Authors have referred to $h(n)$ as the *abundancy* of n, the *index* of n, or the *abundancy index* of n. We have that

$$h(n) = \frac{\sum_{d|n} d}{n} = \sum_{d|n} \frac{d}{n} = \sum_{d|n} \frac{1}{d}. \tag{7}$$

From Equation (7) we have that if $a > 1$, then we have $h(an) > h(n)$. In particular, any multiple of an abundant number is itself abundant, and thus no perfect number can be divisible by an abundant number or be divisible by a smaller perfect number. One can use this, along with Euler's description of an odd perfect number to prove many results; for example it is an easy exercise to use these two facts to prove that no odd perfect number is divisible by 105.

Set

$$H(n) = \prod_{p|n} \frac{p}{p-1}.$$

[3] This observation seems to have been first noted explicitly in the literature in [4].

It is not hard to show that $h(n) \leq H(n)$ with equality if and only if $n = 1$. Also,

$$\lim_{k\to\infty} h(n^k) = H(n).$$

Thus, understanding $h(n)$ is closely connected to understanding $H(n)$.

Much of the work on odd perfect numbers relies on the fundamental observation that we can bootstrap from knowing that a specific prime power divides N to get that other prime powers divide N. For example, if $3^2||N$, then since $\sigma(3^2) = 13$ we may conclude that 13 must also divide N. In general, if $p^k||N$ then we must have $\sigma(p^k)|(2N)$. Note that if p is prime then

$$\sigma(p^k) = 1 + p + p^2 \cdots + p^k = \frac{p^{k+1} - 1}{p - 1}.$$

In general, for any k we have

$$1 + x + x^2 ... + x^k = \frac{x^{k+1} - 1}{x - 1}$$

and we may factor $\frac{x^{k+1}-1}{x-1}$ into a product of cyclotomic polynomials. Thus, a major part of understanding odd perfect numbers comes from understanding the integer values of cyclotomic polynomials. In the context of this paper, much of our work comes from developing a more detailed understanding of the behavior of the cyclotomic polynomials $x^2 + x + 1$ and $x^4 + x^3 + x^2 + x + 1$.

Lemma 1. *Let a and b be distinct odd primes and let p be a prime such that $p|(a^2+a+1)$ and $p|(b^2+b+1)$. If $a \equiv b \equiv 2 \pmod 3$, then $p \leq \frac{a+b+1}{5}$. If $a \equiv b \equiv 1 \pmod 3$ then $p \leq \frac{a+b+1}{3}$.*

This is Lemma 1 from [23]. We will also need the following result, which is Lemma 3 in [16]:

Lemma 2. *Let p, q and r be positive integers. If $p^2+p+1 = r$ and $q^2+q+1 = 3r$, then p is not an odd prime.*

Lemma 3. *If x is a positive integer then the only possible common prime divisor of $x^2 - x + 1$ and $x^6 + 3x^5 + 5x^4 + 6x^3 + 7x^2 + 6x + 3$ is 31.*

Proof. Set $A = x^2 - x + 1$ and $B = x^6 + 3x^5 + 5x^4 + 6x^3 + 7x^2 + 6x + 3$. If $p|(A, B)$ then p divides any linear combination of A and B. In particular, p must divide

$$(5x^5 + 21x^4 + 44x^3 + 58x^2 + 55x + 34)A - (5x + 1)B = 31.$$

So $p = 31$. $\square$

We will also need the following Lemma

Lemma 4. *The only non-negative integer solutions to the equation*

$$x^6 + 3x^5 + 5x^4 + 6x^3 + 7x^2 + 6x + 3 = a^2 + a + 1$$

are $(x, a) = (0, 1)$ *and* $(x, a) = (1, 5)$.

Proof. We can verify by direct computation that these two solutions are the only solutions where $0 \leq 26 \leq x$, so we may assume that $x \geq 27$. Some algebra shows that we may write a in terms of x as

$$a = x^3 + \frac{3}{2}x^2 + \frac{11}{8}x + \frac{7}{16} + t \tag{8}$$

where

$$t = \frac{A}{B},$$

where

$$A = 588x^2 + 876x + 351$$

and

$$B = 256x^3 + 384x^2 + 352x + 240 + 128\sqrt{4x^6 + 12x^5 + 20x^4 + 24x^3 + 28x^2 + 24x + 9}.$$

For any integer x, $x^3 + \frac{3}{2}x^2 + \frac{11}{8}x$ is a rational number whose denominator divides 8, and the next term in Equation (8) is $\frac{7}{16}$, so the only way that a can be an integer is if t is a fraction whose denominator is 16. However, we have that

$$0 < t < \frac{876x^2}{256x^3 + 384x^2} = \frac{219}{128x + 96} < \frac{1}{16}.$$

Here the last inequality on the right is due to $x \geq 27$. Since t is strictly between 0 and $1/16$ it cannot be a fraction with denominator 16, and so there are no more solutions. $\square$

The next lemma then follows almost immediately.

Lemma 5. *The only positive integer solution to*

$$x^6 + 3x^5 + 5x^4 + 6x^3 + 7x^2 + 6x + 3 = a^2 - a + 1$$

is $(x, a) = (1, 6)$.

Proof. This follows from Lemma 4 and noting that this equation is identical to the equation from that lemma but with $a - 1$ substituted for a. $\square$

Lemma 6. *Let* x *be a prime with* $\sigma(x^4) = x^4 + x^3 + x^2 + x + 1$ *prime, and suppose that* $a = \sigma(x^4)$ *is prime. Then* $\sigma(a^2) = a^2 + a + 1$ *has at least two distinct prime factors. Furthermore, if* $\sigma(a^2) = bc$ *for two distinct primes* b *and* c*, then either* $11|\sigma(b^4)$ *or* $11|\sigma(c^4)$*.*

Proof. Assume that x is prime, and assume further that $a = \sigma(x^4)$ is prime. Then $a = x^4 + x^3 + x^2 + x + 1$ and we have $\sigma(a^2) = a^2 + a + 1$. A straightforward calculation gives us

$$\begin{aligned}\sigma(a^2) &= x^8 + 2x^7 + 3x^6 + 4x^5 + 6x^4 + 5x^3 + 4x^2 + 3x + 3 \\ &= (x^2 - x + 1)(x^6 + 3x^5 + 5x^4 + 6x^3 + 7x^2 + 6x + 3).\end{aligned} \tag{9}$$

By Lemma 3 we have that $\sigma(a^2)$ must have at least two distinct prime factors, unless both $(x^2 - x + 1)$ and $x^6 + 3x^5 + 5x^4 + 6x^3 + 7x^2 + 6x + 3$ are powers of 31. But they cannot both be a power of 31. To see why, note that this would make $(x^2 - x + 1)^3$ also a power of 31, and we would have a contradiction due to the fact that as long as $x > 2$ we have

$$(x^2 - x + 1)^3 < x^6 + 3x^5 + 5x^4 + 6x^3 + 7x^2 + 6x + 3 < 31(x^2 - x + 1)^3.$$

To prove the last part of this lemma, we now note that if $\sigma(a^2) = bc$ for two primes b and c we must have that one of the primes is $x^2 - x + 1$ and the other prime is $x^6 + 3x^5 + 5x^4 + 6x^3 + 7x^2 + 6x + 3$. Without loss of generality, let us assume that $b = x^2 - x + 1$ and that $c = x^6 + 3x^5 + 5x^4 + 6x^3 + 7x^2 + 6x + 3$. It is easy to check by looking at x modulo 11 that for any x we have that one of the numbers $b^4 + b^3 + b^2 + b + 1$ or $c^4 + c^3 + c^2 + c + 1$ or $x^4 + x^3 + x^2 + x + 1$ is congruent to 0 modulo 11; since $x^4 + x^3 + x^2 + x + 1$ is prime and not equal to 11, we conclude that either $\sigma(b^4)$ or $\sigma(c^4)$ must be divisible by 11. □

Lemma 7. *There are no odd primes primes x, a, b, c, d satisfying the conditions:*

1. $a = \sigma(x^4)$
2. $\sigma(a^2) = \sigma(b^2)\sigma(c^2)\sigma(d^2)$
3. $\sigma(b^2)$, $\sigma(c^2)$, *and* $\sigma(d^2)$ *are all prime.*

Proof. Assume that we have such a solution. Then by the same logic as in the proof of Lemma 6,

$$\begin{aligned}\sigma(a^2) &= x^8 + 2x^7 + 3x^6 + 4x^5 + 6x^4 + 5x^3 + 4x^2 + 3x + 3 \\ &= (x^2 - x + 1)(x^6 + 3x^5 + 5x^4 + 6x^3 + 7x^2 + 6x + 3).\end{aligned} \tag{10}$$

We have that $\sigma(b^2) = b^2+b+1$, $\sigma(c^2) = c^2+c+1$, and $\sigma(d^2) = d^2+d+1$. Since all three of these quantities are prime we must have one of them equal to either x^2-x+1 or $x^6+3x^5+5x^4+6x^3+7x^2+6x+3$. Without loss of generality, we will assume that this is b^2+b+1. We have two cases: either $b^2+b+1 = x^6+3x^5+5x^4+6x^3+7x^2+6x+3$ or $b^2 + b + 1 = x^2 - x + 1$. The first case is ruled out by Lemma 4 so we must have $b^2 + b + 1 = x^2 - x + 1$. Note that $x^2 - x + 1 = (x - 1)^2 + (x - 1) + 1$. Thus, $b^2 + b + 1 = (x - 1)^2 + (x - 1) + 1$ and so $b = x - 1$ since $t^2 + t + 1$ is a strictly increasing function for $t > 1/2$. But $b = x - 1$ is impossible since b and x are both odd primes. □

Lemma 8. *We cannot have integers a, b, c, d with $a \equiv b \equiv c \equiv d \equiv 1 \pmod 5$ and also satisfying $a^2 + a + 1 = (b^2 + b + 1)(c^2 + c + 1)(d^2 + d + 1)$.*

Proof. This just follows from observing that the left side of the equation is congruent to 3 modulo 5 and the right side is congruent to 2 modulo 5. □

Let (x, a, b, c) be a quadruple of odd primes all greater than 3. We say that they form a *triple threat* if they satisfy two conditions:

1. $x^2 + x + 1 = (a^2 + a + 1)(b^2 + b + 1)(c^2 + c + 1)$;
2. $a^2 + a + 1$, $b^2 + b + 1$, and $c^2 + c + 1$ are all prime.

If we could show that there are no triple threats in general, then we could substantially improve our bounds in this paper for both the $3 \mid N$ case and the $3 \nmid N$ case. However, we are presently unable to do that, and so we must satisfy ourselves with instead proving substantial enough restrictions on what triple threats can look like. We will in particular prove that we cannot have x congruent to 1 modulo 5 while also having one of a, b or c also congruent to 1 modulo 5. Assume that (x, a, b, c) is a triple threat and that $x \equiv a \equiv 1 \pmod 5$. Then without loss of generality one must have

$$(c, d) \in \{(1, 2), (2, 3), (4, 4) \pmod 5\}. \tag{11}$$

We will rule out each of these three options separately. We do so by proving various statements about the equation $x^2 + x + 1 = (a^2 + a + 1)(b^2 + b + 1)p$ with x, a, b, $a^2 + a + 1$, $b^2 + b + 1$ and p all prime. This method of approach has two advantages. First, while triple threats do not seem to exist, solutions of this equation do exist. Thus we will at least be proving statements about actual mathematical objects. Second, this equation appears to be of natural interest for extending results beyond this paper, as will be discussed later.

Lemma 9. *Assume that $x^2 + x + 1 = (a^2 + a + 1)(b^2 + b + 1)p$ with x, $a^2 + a + 1$, $b^2 + b + 1$, a, b and p all primes greater than 3. Assume also that $a \leq b$. Then we have one of four possibilities.*

1. *We have $(a^2+a+1)|(x-a)$, $(b^2+b+1)|(x-b)$, $(x+a+1)|(p(a+b+1)+\frac{x-b}{b^2+b+1})$, and $(x + b + 1)|(p(a + b + 1) + \frac{x-a}{a^2+a+1})$.*
2. *We have $(a^2 + a + 1)|(x - a)$, $(b^2 + b + 1)|(x + b + 1)$, and $(x + a + 1)|(p(b - a) - \frac{x+b+1}{b^2+b+1})$.*
3. *We have $(a^2 + a + 1)|(x + a + 1)$ and $(b^2 + b + 1)|(x - b)$, and $(x + b + 1)|(p(b - a) + \frac{x+a+1}{a^2+a+1})$.*
4. *We have $(a^2 + a + 1)|(x + a + 1)$ and $(b^2 + b + 1)|(x + b + 1)$ and $(x - a)|(p(a + b + 1) - \frac{x+b+1}{b^2+b+1)}$ and $(x - b)|(p(a + b + 1) - \frac{x+a+1}{a^2+a+1})$.*

In all four cases, the quantities on the right-hand sides of the divisibility relations are positive.

Proof. Assume that $x^2+x+1 = (a^2+a+1)(b^2+b+1)p$ with x, a^2+a+1, b^2+b+1, a, b and p all primes greater than 3. Then

$$(x-a)(x+a+1) = x^2+x+1-(a^2+a+1) = (a^2+a+1)(p(b^2+b+1)-1)$$

and

$$(x-b)(x+b+1) = x^2+x+1-(b^2+b+1) = (b^2+b+1)(p(a^2+a+1)-1).$$

Since a^2+a+1 and b^2+b+1 are prime we have four situations:

1. $(a^2+a+1)|(x-a)$ and $(b^2+b+1)|(x-b)$;
2. $(a^2+a+1)|(x-a)$ and $(b^2+b+1)|(x+b+1)$;
3. $(a^2+a+1)|(x+a+1)$ and $(b^2+b+1)|(x-b)$;
4. $(a^2+a+1)|(x+a+1)$ and $(b^2+b+1)|(x+b+1)$.

Note that it is easy to check that we cannot have $a^2+a+1|(x-a, x+a+1)$, and the symmetric remark applies to b^2+b+1. Consider each of these four situations as a separate case.

`Case I.` We have $(a^2+a+1)|(x-a)$ and $(b^2+b+1)|(x-b)$. Set $k_a = \frac{x-a}{a^2+a+1}$ and $k_b = \frac{x-b}{b^2+b+1}$. We have

$$(x+a+1)|(p(b^2+b+1)-1)$$

and

$$(x+b+1)|(p(a^2+a+1)-1).$$

Note that

$$x+a+1 = k_a(a^2+a+1)+2a+1 = k_b(b^2+b+1)+a+b+1$$

and

$$x+b+1 = k_a(a^2+a+1)+a+b+1 = k_b(b^2+b+1)+2b+1.$$

We then have

$$(x+a+1)|(p(x+a+1)-k_b(p(b^2+b+1)-1)).$$

We have that

$$p(x+a+1)-k_b(p(b^2+b+1)-1) = p(k_b(b_2+b+1)+a+b+1)-k_b(p(b^2+b+1)-1).$$

We then note that

$$p(k_b(b_2+b+1)+a+b+1)-k_b(p(b^2+b+1)-1) = p(a+b+1)+k_b = p(a+b+1)+\frac{x-b}{b^2+b+1}.$$

The other relation then follows by symmetry.

Case II. We have $(a^2+a+1)|(x-a)$ and $(b^2+b+1)|(x+b+1)$. Set

$$k_a = \frac{x-a}{a^2+a+1}$$

and

$$k_b = \frac{x+b+1}{b^2+b+1}.$$

We then have that $(x+a+1)|(p(b^2+b+1)-1)$. We have

$$x+a+1 = k_a(a^2+a+1)+2a+1 = k_b(b^2+b+1)+a-b,$$

and

$$(x+a+1)|(k_b(p(b^2+b+1)-1)-p(k_b(b^2+b+1+1)-b+a)) = -(p(b-a)-k_b).$$

We then have $(x+a+1)|(p(b-a)-\frac{x+b+1}{b^2+b+1})$. We need to show that $p(b-a)-\frac{x+b+1}{b^2+b+1} > 0$. Assume that $p(b-a)-k_b \leq 0$. Note in this case we have $b \neq a$, and so $p(b-a) \leq \frac{x+b+1}{b^2+b+1}$. Since $a \equiv b \equiv 2 \pmod 3$, we have that $b-a \geq 6$. Thus, $6p \leq \frac{x+b+1}{b^2+b+1}$, and we get that $2pb^2 < x$. Then

$$2pa^2 < 2pb^2 < x.$$

We then have

$$4p^2a^2b^2 < x^2 < x^2+x+1 = pa^2b^2,$$

which is a contradiction. Thus, $p(b-a)-k_b$ is positive.

Case III. We have that

$$(a^2+a+1)|(x+a+1)$$

and

$$(b^2+b+1)|(x-b).$$

We set $k_a = \frac{x+a+1}{a^2+a+1}$ and $k_b = \frac{x-b}{b^2+b+1}$. We then have from logic identical to that in Case II that $(x+b+1)|(p(a-b)-k_a)$. The right-hand side is negative, so $(x+b+1)|(p(b-a)+k_a)$ is positive.

Case IV. We have

$$(a^2+a+1)|(x+a+1)$$

and

$$(b^2+b+1)|(x+b+1).$$

We then have $(x-a)|(p(x-a)-k_b(p(b^2+b+1)-1)$ where $k_b = \frac{x+b+1}{b^2+b+1}$. From $x-a = k_b(b^2+b+1)-a-b-1$ we get that $(x-a)|(p(a+b+1)-k_b)$. The other divisibility relation follows from similar reasoning. Positivity follows from an argument similar to that of Case II. □

Lemma 10. *Assume that $x^2+x+1 = (a^2+a+1)(b^2+b+1)p$, with x, a^2+a+1, b^2+b+1, a, b and p all primes greater than 3 and $a \le b$. Then $2b < p$.*

Proof. Note that we must have that $x \equiv a \equiv b \equiv 2 \pmod 3$. We consider each of the four cases from Lemma 9.

`Case I.` We have that $(x+a+1)|(p(a+b+1)+k_b)$ where $k_b = \frac{x-b}{b^2+b+1}$. Thus $x+a+1 \le p(a+b+1)+k_b$. This gives us that

$$k_b(b^2+b+1)+a+b+1 \le p(a+b+1)+k_b.$$

This is the same as

$$\frac{k_b b^2}{a+b+1} + \frac{k_b b}{a+b+1} + 1 \le p.$$

Note that $a+b+1 \le 2b$ and that congruence arguments give us that $k_b \ge 6$. We then have that $a+b+1 \le 2b+1 < 3b$. We get that

$$2b+2 < \frac{k_b b^2}{a+b+1} + \frac{k_b b}{a+b+1} + 1 \le p.$$

`Case II.` We have $(x+a+1)|(p(b-a)-k_b)$ with $k_b = \frac{x+b+1}{b^2+b+1}$. Note that $p(b-a)-k_b$ is positive. We have that $p(b-a)-k_b \equiv 1 \pmod 6$, and $x+a+1 \equiv 5 \pmod 6$. Therefore $h(x+a+1) = p(b-a)-k_b$ for some $h \ge 5$. Thus we have

$$5(x+a+1) \le p(b-a)-k_b.$$

Since $k_b = \frac{x+b+1}{b^2+b+1} = \frac{x}{b^2+b+1} + \frac{b+1}{b^2+b+1}$ and $b > 3$, we have that $k_b < \frac{x}{12}$. Thus

$$5x \le p(b-a) - \frac{x}{7}.$$

Then $\frac{36}{5}x \le p(b-a)$. Since $b^2+b+1 \le x$, the desired inequality follows.

`Case III.` Case III is very similar to Case II. We have that $(x+b+1)|(p(b-a)+k_a)$ where $k_a = \frac{x+a+1}{a^2+a+1}$, We have that $x+b+1 \le p(b-a)+k_a$. We have $k_a \le \frac{x}{13}$, and so we have that

$$\frac{12}{13}x \le p(b-a)-b-1.$$

From $(b^2+b+1)|(x-b)$, we get that $b^2+b+1 \le \frac{x}{6}$ which gives us the desired inequality.

`Case IV.` We then have $(x-a)|(p(x-a)-k_b(p(b^2+b+1)-1)$ where $k_b = \frac{x+b+1}{b^2+b+1}$. From $x-a = k_b(b^2+b+1)-a-b-1$ we get that $(x-a)|(p(a+b+1)-k_b)$. Thus $k_b(b^2+b+1)-a-b-1 \leq p(a+b+1)-k_b$ and so we get that from $k_b \geq 5$ and $k_b b^2 < p(a+b+1)$ that $2b < p$. □

We get the following as an immediate corollary.

Corollary 2. *Assume that $x^2+x+1 = (a^2+a+1)(b^2+b+1)p$. Assume that a^2+a+1, b^2+b+1, and $p > 3$ are all prime. Then $p > x^{2/5}$.*

Proof. Assume that $x^2+x+1 = (a^2+a+1)(b^2+b+1)p$, with a^2+a+1, b^2+b+1, and p prime. As usual, assume that $a \leq b$. Then from Lemma 10 we have that $x^2+x+1 = p(a^2+a+1)(b^2+b+1) < p(\frac{p^2}{4}+\frac{p}{2}+1)(\frac{p^2}{4}+\frac{p}{2}+1) \leq p^5$ from which the desired inequality follows. □

Lemma 11. *If $x^2+x+1 = (a^2+a+1)(b^2+b+1)p$, with a^2+a+1, b^2+b+1, a, x and p are all primes prime greater than 3, and $b \geq a \geq 5$, then $a^2+a+1 < \frac{49}{4}p$.*

Proof. This proof requires breaking down the cases above in further detail. In Case I, $x+b+1 \leq p(a+b+1)+k_a$. Note that $k_a \leq \frac{x}{7}$ which yields

$$\frac{6}{7}x \leq p(a+b+1)-b-1.$$

So

$$x \leq \frac{7}{6}(p(a+b+1)-b-1) \leq \frac{7}{6}p(3b)-1 = \frac{7}{2}pb-1.$$

We then have that

$$p(a^2+a+1)b^2 < p(a^2+a+1)(b^2+b+1) = x^2+x+1$$

which together with

$$x^2+x+1 \leq (\frac{7}{2}(pb-1))^2+\frac{7}{2}pb-1+1 < \frac{49}{4}p^2b^2$$

implies that $a^2+a+1 < \frac{49}{4}p$. The other cases are similar. □

Lemma 12. *If $x^2+x+1 = (a^2+a+1)(b^2+b+1)p$, with a^2+a+1, b^2+b+1, and $p > 3$ prime and $b \geq a \geq 3$, then $a^2+a+1 < (\frac{25}{2})^{1/3}x^{2/3} \leq 3x^{2/3}$.*

Proof. We have

$$x^2+x+1 = p(a^2+a+1)(b^2+b+1) \geq \frac{4}{49}(a^2+a+1)^3.$$

Thus,

$$\frac{50}{4}x^2 \geq (a^2+a+1)^3$$

which leads to the desired inequalities. □

Note that this means that we can take $k_a \geq \frac{x^{1/3}}{3}$ in all cases of Lemma 10 and can get tighter versions of that result.

Lemma 13. *Assume that x is a prime. Then $x^2 + x + 1$ is not a perfect cube.*

Proof. Assume that x is prime and $x^2 + x + 1 = k^3$. Then we have that

$$x(x+1) = k^3 - 1 = (k-1)(k^2 + k + 1).$$

Since x is prime, either $x|(k-1)$ or $x|(k^2+k+1)$. If $x|(k-1)$, then $x \leq k-1$ and $k^3 \geq (x+1)^3 > x^2 + x + 1 = k^3$, which is impossible. Now consider the possibility that $x|(k^2 + k + 1)$. Since x is prime, then either $x = 3$, which does not give a solution, or $x \equiv 1 \pmod 3$. Thus, $x^2 + x + 1 \equiv 3 \pmod 9$, but no perfect cube is congruent to 3 modulo 9 and so we again reach a contradiction. □

Lemma 14. *Assume that $x^2 + x + 1 = (a^2 + a + 1)(b^2 + b + 1)p$, with $a^2 + a + 1$, $b^2 + b + 1$, a, and b all primes greater than 3. Assume that $b > 5$. Further assume that $x \equiv a \equiv b \equiv 1$ (mod 5). Then $\frac{11}{3}b < p$.*

Proof. This follows the same sort of logic as Lemma 10.

`Case I.` We get that

$$\frac{k_b b^2}{a+b+1} + \frac{k_b b}{a+b+1} + 1 \leq p.$$

Note that congruence arguments give us that $k_b \geq 6$ and thus $a+b+1 \leq 2b+1 < 3b$. We have by congruence arguments that $k_b \geq 30$, and so $10b < p$.

`Case II.` We have as before $\frac{34}{35}x \leq p(b-a)$ and $x + b + 1 = k_b(b^2 + b + 1)$. Note that k_b is odd. We have that $x + b + 1 \equiv 2 \pmod 3$ and $b^2 + b + 1 \equiv 1 \pmod 3$, so $k_b \equiv 2 \pmod 3$. Similarly, we have that $k_b \equiv 1 \pmod 5$. So $k_b \geq 11$. We then have

$$b^2 + b + 1 \leq \frac{x+b+1}{11}.$$

This is the same as

$$11b^2 + 10b + 10 \leq x,$$

and so

$$11b^2 + 10b + 10 \leq \frac{35}{34}p(b-a),$$

which is stronger than the desired inequality.

`Case III.`We have $(12/13)x \leq p(b-a) - b - 1$ and $k_b = \frac{x-b}{b^2+b+1}$. We then have $30|k_b$, and so

$$b^2 + b + 1 \leq \frac{x-b}{30}.$$

Thus, $30b^2 + 31b + 30 \leq x$, and combining as before yields an inequality stronger than the one desired.

Case IV. We have that $k_b = \frac{x+b+1}{b^2+b+1}$ as in Case II. As in Case II, we get that k_b is odd, $k_b \equiv 2 \pmod 3$ and $k_b \equiv 1 \pmod 5$. So $k_b \geq 11$. In Case IV we had that $k_b b^2 < p(a+b+1)$ and since $a+b+1 < 3b$ this becomes

$$11b^2 < 3pb$$

which implies that

$$\frac{11}{3}b < p.$$

□

Lemma 15. *Suppose that x, a, b, and p are all primes greater than 3 where*

$$x^2 + x + 1 = (a^2 + a + 1)(b^2 + b + 1)p.$$

Suppose also that $a^2 + a + 1$ and $b^2 + b + 1$ are prime. Suppose further that $a \leq b$. Then in Case II and Case III we have $a^2 + a + 1 < \frac{p}{16}$.

Proof. Assume we are in Case II. So we have $(x + a + 1)|(p(b - a) - \frac{x+b+1}{b^2+b+1})$. We note that $x + a + 1 \equiv 2 \pmod 3$ and $p(b-a) - \frac{x+b+1}{b^2+b+1} \equiv 1 \pmod 3$. Note also that both quantities are odd. We thus have $m(x + a + 1) = p(b - a) - \frac{x+b+1}{b^2+b+1}$ for some $m \geq 5$. We then have

$$5(x + a + 1) \leq p(b - a) - \frac{x + b + 1}{b^2 + b + 1}.$$

We then have $x < \frac{pb}{5} - 2$. Then

$$(a^2 + a + 1)(b^2 + b + 1)p = x^2 + x + 1 = (\frac{pb}{5} - 2)^2 + \frac{pb}{5} - 2 + 1 < \frac{p^2b^2}{25}.$$

Therefore we have,

$$(a^2 + a + 1)(b^2 + b + 1)p < \frac{p^2b^2}{25}$$

and so we have that $a^2 + a + 1 < \frac{p}{25}$ which implies the desired result.

Now for Case III: We have that $(x + b + 1)|(p(b - a) + \frac{x+a+1}{a^2+a+1})$. By similar logic as above we have that $m(x + b + 1) = p(b - a) + \frac{x+a+1}{a^2+a+1}$ where $m \equiv 5 \pmod 6$. We have $5(x + b + 1) \leq p(b - a) + \frac{x+a+1}{a^2+a+1}$ and so

$$x < \frac{pb}{4} - 2.$$

Thus,

$$(a^2 + a + 1)(b^2 + b + 1)p = x^2 + x + 1 = \left(\frac{pb}{4} - 2\right)^2 + \frac{pb}{4} - 2 + 1 < \frac{p^2b^2}{16},$$

from which the result follows. □

Lemma 16. *Suppose that x, a, b, and p are all primes greater than 3 where*

$$x^2 + x + 1 = (a^2 + a + 1)(b^2 + b + 1)p.$$

Suppose that $a^2 + a + 1$ and $b^2 + b + 1$ are prime. Suppose also that $a \leq b$. Assume further that we have Case IV and that we have $x - a = p(a + b + 1) - k_b$ and $x - b = p(a + b + 1) - k_a$. Then we must have $a = b$ and $(a^2 + a + 2)|(7(p + 1))$.

Proof. Assume as given. We then have

$$(x - b)(a^2 + a + 1) + x + a + 1 = p(a + b + 1)(a^2 + a + 1), \tag{12}$$

and

$$(x - a)(b^2 + b + 1) + x + b + 1 = p(a + b + 1)(b^2 + b + 1). \tag{13}$$

Assume for now that $a \neq b$. from Equations (12) and (13) we have that

$$\frac{(x - b)(a^2 + a + 1) + x + a + 1}{a^2 + a + 1} = p(a + b + 1) = \frac{(x - a)(b^2 + b + 1) + x + b + 1}{b^2 + b + 1}. \tag{14}$$

We can solve the above for x to get that

$$x = \frac{(b - a)(a^2 + a + 1)(b^2 + b + 1) - (b + 1)(a^2 + a + 1) + (a + 1)(b^2 + b + 1)}{b^2 + b - a^2 - a}. \tag{15}$$

Now, it turns out the top and bottom both have a factor of $b - a$ and this is a meaningful solution for x because we have assumed that $b \neq a$. Simplifying we get that

$$x = \frac{a^2b^2 + a^2b + a^2 + ab^2 + 2ab + 2a + b^2 + 2b + 1}{a + b + 1},$$

and this forces x to be even which is a contradiction since x is an odd prime. We may thus assume that $a = b$.

Thus, Equation (12) and Equation (13) become the same thing:

$$(x - a)(a^2 + a + 1) + x + a + 1 = p(2a + 1)(a^2 + a + 1). \tag{16}$$

We may rearrange Equation (16) to obtain

$$x(a^2 + a + 2) = (p(2a + 1) + a)(a^2 + a + 1) - a - 1, \tag{17}$$

$$(a^2 + a + 2)|((p(2a + 1) + a)(a^2 + a + 1) - a - 1). \tag{18}$$

We also trivially have

$$(a^2 + a + 2)|((p(2a + 1) + a)(a^2 + a + 2)). \tag{19}$$

We may then conclude that a^2+a+2 divides the difference of Equation (18) and Equation (19). So a^2+a+2 divides $p(2a+1)+2a+1=(p+1)(2a+1)$. It is easy to check that $\gcd(a^2+a+2,2a+1)|7$. Thus, we have

$$(a^2+a+2)|(7(p+1)). \tag{20}$$

which was what was claimed. $\square$

Lemma 17. *Suppose that x, a, b and p are all primes greater than 3 where*

$$x^2+x+1=(a^2+a+1)(b^2+b+1)p.$$

Suppose further that $x \equiv a \equiv b \equiv 1 \pmod 5$ and that a^2+a+1 and b^2+b+1 are prime. Suppose also that $a \le b$, and $p \ge 47$. Then we have $a^2+a+1<p$.

Proof. We will break into four cases in the same way as before. Cases II and III are handled by Lemma 15. In this situation, almost all the serious work will be in Case IV. We will handle Case I and then Case IV. Case I: We have as before that $(x+b+1)|(p(a+b+1)+k_a)$ where $k_a=\frac{x-a}{a^2+a+1}$. Set $m(x+b+1)=p(a+b+1)+k_a$. We have that $x+b+1 \equiv 3 \pmod 5$ and that $p(a+b+1)+k_a \equiv 1 \pmod 5$. Thus, we have $m \equiv 2 \pmod 5$. Similarly, we have that $m \equiv 1 \pmod 3$. So $m \ge 7$ and so $7(x+b+1) \le p(a+b+1)+k_a$. Since $a^2+a+1>7$, we have that $k_a \le \frac{x}{7}$, so

$$\frac{48}{7}x \le p(a+b+1)-7b-7$$

which yields

$$x \le \frac{7}{48}p(a+b+1)-\frac{49}{48}b-\frac{49}{48}.$$

We have that $a+b+1<3b$ and so

$$x \le \frac{7}{16}pb-\frac{49}{48}-\frac{49}{48} \le \frac{7}{16}pb-2.$$

We then have

$$(a^2+a+1)(b^2+b+1)p=x^2+x+1 \le (\frac{7}{16}pb-2)^2+\frac{7}{16}pb-2+1<\left(\frac{7}{16}\right)^2(pb^2).$$

From the above we get that $a^2+a+1 \le \left(\frac{7}{16}\right)^2 p<p$.

`Case IV.` We have $(x-a)m_a=p(a+b+1)-k_b$ and $(x-b)m_b=p(a+b+1)-k_a$ for some m_a and m_b.

First, let us consider the situation where $m_a=m_b=1$. By Lemma 16 we have that $k(a^2+a+2)=7(p+1)$ for some k. Note that $a^2+a+2 \equiv 4 \pmod 5$. We also have $7(p+1) \equiv 1 \pmod 5$. Thus, we have $k \equiv 4 \pmod 5$. Similarly, we have that $k \equiv 1 \pmod 3$. So $k \equiv 4 \pmod{15}$. Consider the possibility of $k = 4$. If that is the case, then we have $4(a^2 + a + 2) = 7(p + 1)$, and this is the same as

$$7p = 4a^2 + 4a + 1 = (2a + 1)^2.$$

But then we must have $p = 7$ and $a = 3$, which is ruled out by our initial assumption that $a > 3$. Thus, since $k \equiv 4 \pmod{15}$, we get that $k \geq 19$. In that case we have $19(a^2 + a + 1) \leq 7(p + 1)$, from which it follows that $a^2 + a + 1 < p$. Next we will consider $m_a = 2$ and $m_b = 1$. We need to consider then the equation

$$2(x - a) = p(a + b + 1) - k_b.$$

Since $m_b \neq m_a$ we have that $a \neq b$, and thus $a \leq b - 30$ (since a and b agree modulo 30). So we have that

$$x = \frac{p(a + b + 1)}{2} - \frac{k_b}{2} + a \leq \frac{p(2b - 29)}{2} - \frac{k_a}{2} + a.$$

We note that Lemma 10 implies that $a < p$, and so we have that

$$x \leq pb - \frac{29p}{2} + p < pb - 2,$$

from which the same logic as used in Case I holds. The case when $m_b = 2$ and $m_a = 1$ is nearly identical, as is the case when either m_a or m_b being 2 but with $a \neq b$. We will then next consider the case $m_a = m_b = 2$, and $a = b$. We have

$$2(x - a) = p(2a + 1) - \frac{x + a + 1}{a^2 + a + 1}. \tag{21}$$

Solving for x we obtain

$$(2a^2 + 2a + 3)x = (p(2a + 1) + 2a)(a^2 + a + 1) - a - 1. \tag{22}$$

Equations (22) along with the fact that $x^2 + x + 1 = (a^2 + a + 1)^2 p$ together imply that $p = \frac{4a^4+8a^3+8a^2+8a+7}{(2a+1)^2}$. Thus, we have $(2a + 1)^2 | (4a^4 + 8a^3 + 8a^2 + 8a + 7)$ but $4a^4 + 8a^3 + 8a^2 + 8a + 7 = (2a + 1)^2(a^2 + a + 1) + (-a^2 - a + 6)$. Thus, $(2a + 1)^2 | (a^2 + a - 6)$ which is impossible.

By the above remarks we must have either $m_a \geq 3$ or $m_b \geq 3$. We will consider $m_a \geq 3$ (the case for $m_b \geq 3$ is essentially identical). We have

$$3(x - a) \leq p(a + b + 1) - k_b.$$

We note that $a + b + 1 \leq 2b + 1$, and thus

$$x \leq \frac{p(2b + 1)}{3} - k_b + a.$$

We note that $p \geq 47$, together with Lemma 11, gives us that $a < \frac{p}{3}$. We again obtain that

$$x \leq pb - 2,$$

and again draw the same conclusion. $\square$

Lemma 18. *Suppose that we have odd primes x, a, b, p all primes greater than 3 where*

$$x^2 + x + 1 = (a^2 + a + 1)(b^2 + b + 1)p.$$

Suppose further that $x \equiv a \equiv 1 \pmod{5}$ and that $b \equiv 2 \pmod{5}$. Suppose that $a^2 + a + 1$ and $b^2 + b + 1$ are prime. Suppose that $a \leq b$, and $p \geq 47$. Then we have $a^2 + a + 1 < p$.

Proof. As usual, we will have four cases to check, and as in the last lemma most of the difficulty will be in Case IV. In this lemma, as in the last one, we will be able to rely on the previous results concerning this situation. However, due to the different congruence assumption we cannot here make use of Lemma 17. Note that in this lemma we now have $p \equiv 3 \pmod{5}$, rather than in the previous lemma where we had $p \equiv 2 \pmod{5}$.

`Case I.` We have as before that $(x+b+1)|(p(a+b+1)+\frac{x-a}{a^2+a+1})$. Set $m(x+b+1) = p(a+b+1)+\frac{x-a}{a^2+a+1}$. We have that $p(a+b+1)+\frac{x-a}{a^2+a+1} \equiv 2 \pmod{5}$ and $x+b+1 \equiv 4 \pmod{5}$. Thus $m \equiv 3 \pmod{5}$. Note that m is odd, and also that $(3, m) = 1$, and so $m \geq 13$. We then have

$$13(x + b + 1) \leq p(a + b + 1) + \frac{x - a}{a^2 + a + 1}.$$

The same logic as in the previous lemma then applies. Lemma 15 handles Case II and Case III.

For Case IV we have that $(a^2+a+1)|(x+a+1)$ and $(b^2+b+1)|(x+b+1)$. Note that unlike in Lemma 17 we have immediately that $a \neq b$ because they disagree modulo 5. We have $(x - a)|(p(a + b + 1) - k_b)$ and $(x - b)|(p(a + b + 1) - k_a)$ where $k_b = \frac{x+b+1}{b^2+b+1}$, and $k_a = \frac{x+a+1}{a^2+a+1}$. We have $(x - a)m_a = p(a + b + 1) - k_b$ and $(x-b)m_b = p(a+b+1)-k_a$ for some positive integers m_a and m_b. We will consider various possible options for the pair (m_a, m_b). The pair $(1, 1)$ is already ruled out since then Lemma 16 would force $a = b$.

Since we have that $a \neq b$, it follows that $a \leq b - 6$ (since $a < b$ and $a \equiv b + 1 \pmod{30}$). So we have that

$$x = \frac{p(a + b + 1)}{2} - \frac{k_b}{2} + a \leq \frac{p(2b - 6)}{2} - \frac{k_a}{2} + a.$$

Note that Lemma 10 implies that $a < p$, and so we have that

$$x \leq pb - 3p + p < pb - 2,$$

from which the same logic as used in Case I holds. For the remaining possible options for (m_a, m_b) we follow logic that is essentially identical to those in remaining parts of the Case IV of the proof of Lemma 17. $\square$

Using nearly identical logic to the above Lemma we obtain the following Lemma.

Lemma 19. *Suppose that we have odd primes x, a, b, p, all greater than 3, where*

$$x^2 + x + 1 = (a^2 + a + 1)(b^2 + b + 1)p.$$

Suppose further that $a \equiv 2 \pmod 5$, $x \equiv b \equiv 1 \pmod 5$. Assume that a^2+a+1 and b^2+b+1 are prime. Finally, assume that $a \le b$, and $p \ge 47$. Then $a^2+a+1 < p$.

Lemma 20. *There are no solutions to the equation*

$$x^2 + x + 1 = (p^2 + p + 1)(q^2 + q + 1)(r^2 + r + 1)$$

with x, p, q, r, p^2+p+1, q^2+q+1, r^2+r+1 all prime and with $x \equiv p \equiv q \equiv 1 \pmod 5$. That is, there does not exist any triple threat (x, p, q, r) where $x \equiv p \equiv q \equiv 1 \pmod 5$.

Proof. Assume we have a solution. We note that we must also have $r \equiv 2 \pmod 5$. We must have $\min(p^2 + p + 1, q^2 + q + 1, q^2 + q + 1) > 47$. If any were not, we could use Corollary 2 to conclude that we have $47 > x^{2/5}$, this gives us only a finite set of x to check and we can easily verify that none of them are solutions. We may without loss of generality also assume that $p \le q$. We use Lemma 17 to conclude that $p^2 + p + 1 < r^2 + r + 1$, and so $p < r$. We may apply Lemma 18 to get that $p < q$. We have two cases to consider. Either $p < r < q$ or $p < q < r$. If $p < r < q$, then Lemma 18 gives us that $r^2 + r + 1 < p^2 + p + 1$ and hence $r < p$ which is impossible. So we may assume that $p < q < r$, but then by Lemma 19 we have that $q^2 + q + 1 < p^2 + p + 1$ and hence $q < p$ which is impossible. So all cases have lead to a contradiction. $\square$

The basic thrust of the next set of results is very similar.

Lemma 21. *Suppose that $x^2 + x + 1 = (a^2 + a + 1)(b^2 + b + 1)p$ where x, a, b, $a^2 + a + 1$, $b^2 + b + 1$ and p are odd primes greater than 3. Suppose further that $p = c^2 + c + 1$ for some c where $c \equiv 5 \pmod 6$. Suppose that $a \le b$ and that we are in Case I. Then $a^2 + a + 1 < \frac{p}{4}$.*

Proof. Assume as given. So we have $(x + a + 1)|(p(a + b + 1) + \frac{x-b}{b^2+b+1})$ and $(x + b + 1)|(p(a + b + 1) + \frac{x-a}{a^2+a+1})$. We may set $m_a(x + a + 1) = p(a + b + 1) + \frac{x-b}{b^2+b+1}$ and $m_b(x + b + 1) = p(a + b + 1) + \frac{x-a}{a^2+a+1}$ We will first assume that $m_a = m_b = 1$ and then handle the remaining cases. If $m_a = m_b = 1$, then we have

$$x + a + 1 = p(a + b + 1) + \frac{x - b}{b^2 + b + 1} \tag{23}$$

and
$$x+b+1=p(a+b+1)+\frac{x-a}{a^2+a+1} \tag{24}$$
We will first assume that $a \neq b$, and arrive at a contradiction. We will then handle $a=b$ (which is where we will need the assumption that $p=c^2+c+1$). Assume that $a \neq b$. Then we may subtract Equation (23) from Equation (24) to get that
$$b-a=\frac{x-a}{a^2+a+1}-\frac{x-b}{b^2+b+1}.$$
This is the same as
$$(a^2+a+1)(b^2+b+1)(b-a)=x(b-a)(a+b+1)+ab(b-a)+(b-a).$$
Since $b \neq a$ we have $b-a \neq 0$ and so we may divide by $b-a$ to get
$$(a^2+a+1)(b^2+b+1)=x(a+b+1)+ab+1.$$
This is the same as $x=\frac{a^2b^2+a^2b+b^2a+a^2+b^2+a+b}{a+b+1}$. But we must have $a \equiv b \equiv 2 \pmod 3$ and also $x \equiv 2 \pmod 3$. But $a \equiv b \equiv 2 \pmod 3$ forces the right-hand side of the above to be congruent to 1 modulo 3. So we must have $a=b$. Since $a=b$ we have
$$x+a+1=p(2a+1)+\frac{x-a}{a^2+a+1},$$
which can be rearranged to
$$(p-1)(2a^2+3a^2+3a+1)=-a^3+a^2(x-a)+ax.$$
We have that $(a, 2a^2+3a^2+3a+1)=1$ and so $a|(p-1)$. Now since $p=c^2+c+1$ this is the same as saying that $a|(c(c+1))$. Either $a|c$ or $a|(c+1)$. If $a|c$, then either $a=c$, or $a<c$. If $a=c$, then $p=a^2+a+1$ and $x^2+x+1=(a^2+a+1)^3$. But this would contradict Lemma 13. If $a<c$, then since $a|c$ and $a \equiv c \equiv 5 \pmod 6$, we would have $7a \leq c$, from which it easily follows that $a^2+a+1<\frac{p}{4}$. If we have $a|(c+1)$, then since $6|(c+1)$, we have that $6a|c+1$ and so $6a \leq c+1$, from which it easily follows that $a^2+a+1<\frac{p}{4}$. We now need to handle the case when m_a and m_b are not both equal to 1. We will look at the case when $m_a \neq 1$ ($m_b \neq 1$ is essentially identical). We have that $m_a(x+a+1)=p(a+b+1)+\frac{x-b}{b^2+b+1}$ for some $m_a>1$. We note that we cannot have m_a even because the right-hand side of the equation is odd. We also cannot have $m_a=3$ because the right-hand side is congruent to 2 modulo 3. We therefore have $m_a \geq 5$. We then have : $5(a+x+1) \leq p(a+b+1)+\frac{x-b}{b^2+b+1}$ from which the desired inequality follows. $\square$

We would like in Lemma 21 to remove the need for the assumption that $p=c^2+c+1$ but for our results here that is not necessary.

Lemma 22. *Suppose that $x^2+x+1=(a^2+a+1)(b^2+b+1)p$ where x, a, b , p, a^2+a+1, and b^2+b+1 are all prime. Assume that $x \equiv a \equiv 1 \pmod 5$ and that $b \equiv 4 \pmod 5$. Assume that $a < b$. Finally, assume that $p = c^2+c+1$ for some $c \equiv 5 \pmod 6$. Then $a^2+a+1 < \frac{p}{4}$.*

Proof. Note that $p \equiv 1 \pmod 5$. We again split into four cases. Case I is handled by Lemma 21. As usual, Cases II and III are handled by Lemma 15. So we need only concern ourselves with Case IV.

In Case IV we have: $(x-b)|(p(a+b+1)-\frac{x+a+1}{a^2+a+1})$. We then have that there exists an m_b such that

$$m_b(x-b) = p(a+b+1) - \frac{x+a+1}{a^2+a+1}.$$

Note that $x-b \equiv 2 \pmod 5$, and $p(a+b+1) - \frac{x+a+1}{a^2+a+1} \equiv 0 \pmod 5$. So $m_b \geq 5$. We then have that $5(x-b) \leq p(a+b+1) - \frac{x+a+1}{a^2+a+1}$ from which the desired bound follows. □

Using nearly identical logic we have the following lemma.

Lemma 23. *Suppose that $x^2+x+1=(a^2+a+1)(b^2+b+1)p$ where x, a, b, p, a^2+a+1 and b^2+b+1 are all prime. Assume that $x \equiv b \equiv 1 \pmod 5$, $a \equiv 4 \pmod 5$, and $a < b$. Assume also that $p = c^2+c+1$ for some $c \equiv 5 \pmod 6$. Then $a^2+a+1 < \frac{p}{4}$.*

Lemma 24. *Suppose that $x^2+x+1=(a^2+a+1)(b^2+b+1)p$ where x, a, b, p, a^2+a+1 and b^2+b+1 are all prime. Assume that $x \equiv 1 \pmod 5$, $a \equiv b \equiv 4 \pmod 5$, $p \equiv 3 \pmod 5$, and that $a < b$. Then $a^2+a+1 < \frac{p}{16}$.*

Proof. We have our four cases as usual with Cases II and III handled by Lemma 15. In Case I we have as before that $(x+b+1)|(p(a+b+1)+\frac{x-a}{a^2+a+1})$. Set $m(x+b+1) = p(a+b+1) + \frac{x-a}{a^2+a+1}$. We have that $x+b+1 \equiv 1 \pmod 5$ and $p(a+b+1) + \frac{x-a}{a^2+a+1} = 4 \pmod 5$. So $m \equiv 4 \pmod 5$. Since $(6,m)=1$ we have that $m \geq 19$. We then have that

$$19(x+b+1) \leq p(a+b+1) + \frac{x-a}{a^2+a+1}$$

from which the desired inequality follows.
In Case IV we have $(x-b)|(p(a+b+1)-\frac{x+a+1}{a^2+a+1})$. We set $m(x-b) = p(a+b+1) - \frac{x+a+1}{a^2+a+1}$. We note that $x-b \equiv 2 \pmod 5$, and $p(a+b+1) - \frac{x+a+1}{a^2+a+1} \equiv 1 \pmod 5$. Thus, $m \equiv 2 \pmod 5$ and so $m \geq 7$. We then have that $7(x-b) \leq p(a+b+1) - \frac{x+a+1}{a^2+a+1}$ from which the desired inequality follows. □

Lemma 25. *There are no solutions to the equation*

$$x^2+x+1=(p^2+p+1)(q^2+q+1)(r^2+r+1)$$

with x, p, q, r, p^2+p+1, q^2+q+1, r^2+r+1 all prime and with $x \equiv p \equiv 1$ (mod 5) and $q \equiv r \equiv 4$ (mod 5). That is, there does not exist any triple threat (x,p,q,r) where $x \equiv p \equiv 1$ (mod 5) and $q \equiv r \equiv 4$ (mod 5).

Proof. Assume we have such a solution. Without loss of generality we may assume that $q \le r$. We then have from Lemma 24 that $q < p$. We thus have either $q < p < r$ or $q < r < p$ (we cannot have $p = r$ since they disagree modulo 5). If we have that $q < p < r$, then from Lemma 22 we have that $p < q$ which is a contradiction. If we have $q < r < p$, then by Lemma 23 we have that $r < q$ which is a contradiction. Since all possibilities lead to a contradiction, the corresponding type of triple threat cannot exist. □

We now turn our attention to our final type of triple threat.

Lemma 26. *Suppose that $x^2+x+1 = (a^2+a+1)(b^2+b+1)p$ where x, a, b, a^2+a+1, b^2+b+1 and p are primes. Suppose further that $x \equiv b \equiv 1$ (mod 5), $a \equiv 3$ (mod 5), and $p \equiv 2$ (mod 5). Suppose that $a < b$. Then we have that $a^2+a+1 < p$.*

Proof. We split into four cases as usual with cases II and III handled by Lemma 15.

`Case I.` We have that $(x+b+1)|(p(a+b+1)+\frac{x-b}{b^2+b+1})$. We have that $m(x+b+1) = p(a+b+1)+\frac{x-b}{b^2+b+1}$ for some m. We note that $x+b+1 \equiv 3 \pmod 5$ and $p(a+b+1)+\frac{x-b}{b^2+b+1} \equiv 1 \pmod 5$. We then have that $m \equiv 2 \pmod 5$. Since m is odd, we have that $m \ge 7$, and $7(x+b+1) \le p(a+b+1)+\frac{x-b}{b^2+b+1}$, from which the desired inequality follows.

`Case IV.` We have that $(x-a)|(p(a+b+1)-\frac{x+b+1}{b^2+b+1})$. We have that for some m, $m(x-a) = p(a+b+1)-\frac{x+b+1}{b^2+b+1}$. We have that $x-a \equiv 3 \pmod 5$, and $p(a+b+1)-\frac{x+b+1}{b^2+b+1} \equiv 4 \pmod 5$. We then have that $m \equiv 3 \pmod 5$. We then have that $3(x-b) \le p(a+b+1)-\frac{x+b+1}{b^2+b+1}$ from which the inequality follows. □

Using nearly identical logic we have the following lemma.

Lemma 27. *Suppose that $x^2+x+1 = (a^2+a+1)(b^2+b+1)p$ where x, a, b, a^2+a+1, b^2+b+1 and p are primes. Suppose further that $x \equiv a \equiv 1$ (mod 5), $b \equiv 3$ (mod 5), and $p \equiv 2$ (mod 5). Suppose that $a < b$. Then we have that $a^2+a+1 < p$.*

Using similar logic we also have the next lemma.

Lemma 28. *Suppose that $x^2+x+1 = (a^2+a+1)(b^2+b+1)p$ where x, a, b, a^2+a+1, b^2+b+1 and p are primes. Suppose further that $x \equiv 1$ (mod 5), $a \equiv 2$ (mod 5), $b \equiv 3$ (mod 5), and $p \equiv 3$ (mod 5). Suppose that $a < b$. Then we have that $a^2+a+1 < p$.*

Proof. We again split into four cases and handle cases II and III via Lemma 15.

`Case I.` We have that $(x+a+1)|(p(a+b+1)+\frac{x-b}{b^2+b+1})$. We have that $x+a+1 \equiv 4 \pmod 5$, and $p(a+b+1)+\frac{x-b}{b^2+b+1} \equiv 1 \pmod 5$. The rest of the case follows as usual.

`Case IV.` We have that $(x-a)|(p(a+b+1)-\frac{x+b+1}{b^2+b+1})$. We have that $x-a \equiv 4 \pmod 5$, and that $p(a+b+1)-\frac{x+b+1}{b^2+b+1} \equiv 3 \pmod 5$, and the rest of the argument follows as usual. □

By nearly identical logic we have the following lemma.

Lemma 29. *Suppose that $x^2+x+1 = (a^2+a+1)(b^2+b+1)p$ where x, a, b, a^2+a+1, b^2+b+1 and p are primes. Suppose further that $x \equiv 1 \pmod 5$, $a \equiv 3 \pmod 5$, $b \equiv 1 \pmod 5$, and $p \equiv 2 \pmod 5$. Suppose that $a < b$. Then we have that $a^2+a+1 < p$.*

Proof. We again split into four cases and handle cases II and III via Lemma 15.
Case I: we have that $(x+a+1)|(p(a+b+1)+\frac{x-b}{b^2+b+1})$. We have that $x+a+1 \equiv 4 \pmod 5$, and $p(a+b+1)+\frac{x-b}{b^2+b+1} \equiv 1 \pmod 5$. The rest of the case follows as usual.

`Case IV.` We have that $(x-a)|(p(a+b+1)-\frac{x+b+1}{b^2+b+1})$. We have that $x-a \equiv 4 \pmod 5$ and that $p(a+b+1)-\frac{x+b+1}{b^2+b+1} \equiv 3 \pmod 5$. The rest of the argument follows as usual. □

We are now in a position where we may prove the following lemma.

Lemma 30. *There are no solutions to the equation*

$$x^2+x+1 = (p^2+p+1)(q^2+q+1)(r^2+r+1)$$

with x, p, q, r, p^2+p+1, q^2+q+1, r^2+r+1 all prime and with $x \equiv p \equiv 1 \pmod 5$ and $q \equiv 2 \pmod 5$, and $r \equiv 3 \pmod 5$. That is, there does not exist any triple threat (x,p,q,r) where $x \equiv p \equiv 1 \pmod 5$ and $q \equiv 2 \pmod 5$, $r \equiv 3 \pmod 5$.

Proof. Assume we have such an (x,p,q,r). Either $p < q$ or $q < p$ (they cannot be equal since they disagree modulo 5). First, let us consider the case that $p < q$. Then by Lemma 18, we have that $p < r$. We then have either $p < q < r$ or $p < r < q$. Let us first consider the case when $p < q < r$. We then have by Lemma 28, that $q < p$ which is a contradiction. Let us then consider the case $p < r < q$. We may then use Lemma 29 to conclude that $p < r$ which is a contradiction. Thus, both of the possibilities for $p < q$ lead to a contradiction. We thus must have $q < p$. From $q < p$ and Lemma 19 we me must have $q < r$. Thus we have either $q < p < r$ or $q < r < p$. If we have $q < p < r$, then by Lemma 29, we have that $p < q$. If $q < r < p$ we may use Lemma 29 to get that $r < q$ and so we have a contradiction. So in each situation we have a contradiction and so the intended type of triple threat does not exist. □

Lemma 31. *If (x,a,b,c) is a triple threat with $x \equiv 1$ (mod 5) then none of a,b or c may be 1 (mod 5).*

Proof. We can enumerate all possible triple threats modulo 5 with $x \equiv 1$ (mod 5). At least one of a, b or c is congruent to 1 modulo 5. We then see that, up to relabeling of the variables, such a triple threat must be one of the forms ruled out by Lemma 20, Lemma 25, or Lemma 30. □

Lemma 32. *Suppose that a and c are distinct odd primes, with a^2+a+1 prime. Assume further that $(a^2+a+1)|(c^2+c+1)$. Then $a^2+a+1 < \frac{c}{2}$. Furthermore, if $(3, c^2+c+1) = 1$, then $a^2+a+1 < \frac{2}{9}c$.*

Proof. Assume as given. Since $c^2+c+1 \equiv 0 \pmod{a^2+a+1}$, and a^2+a+1 is prime, we must have either $c \equiv a \pmod{a^2+a+1}$ or $c \equiv a^2 \pmod{a^2+a+1}$ (since $(a^2+a+1)|((c-a)(a+c+1)))$. We have $c \neq a$ by assumption. We also have that $c \neq a^2$ since c is prime. We then note that $a+(a^2+a+1)$ and $a^2+(a^2+a+1)$ are both even and so c cannot be equal to either. Thus, we have that

$$c \geq a + 2(a^2+a+1) > 2(a^2+a+1)$$

from which the result follows.

Now, under the additional assumption that $(3, c^2+c+1) = 1$, we must have either $c = 3$ (which immediately leads to a contradiction), or we must have $c \equiv 2 \pmod 3$. Since a^2+a+1 is prime, we must have $a \equiv 2 \pmod 3$. Because $a+2(a^2+a+1) \equiv 1 \pmod 3$, we have $c \neq a+2(a^2+a+1)$. Similarly, $a^2+2(a^2+a+1) \equiv 0 \pmod 3$, so $c \neq a^2+2(a^2+a+1)$. We can rule out the next two possible values for c, $a+3(a^2+a+1)$ and $a^2+3(a^2+a+1)$, since they are both even. Then since $a+4(a^2+a+1) \equiv 0 \pmod 3$, this is also not an acceptable value of c either, and so

$$c \geq a^2 + 4(a^2+a+1) > \frac{9}{2}(a^2+a+1)$$

which is the desired inequality. □

Lemma 33. *There are no odd primes a,b,c, with a^2+a+1 and b^2+b+1 prime, and satisfying*

$$c^2+c+1 = 3(a^2+a+1)(b^2+b+1).$$

Proof. Assume we have odd primes a,b,c, with both a^2+a+1 and b^2+b+1 prime, and satisfying

$$c^2+c+1 = 3(a^2+a+1)(b^2+b+1).$$

We may then apply Lemma 32 twice to conclude that

$$a^2 + a + 1 < \frac{c}{2}$$

and that

$$b^2 + b + 1 < \frac{c}{2}.$$

We then have

$$c^2 + c + 1 = 3(a^2 + a + 1)(b^2 + b + 1) < 3\left(\frac{c}{2}\right)\left(\frac{c}{2}\right) = \frac{3c^2}{4} < c^2 + c + 1$$

which is a contradiction. □

Lemma 34. *Suppose that a and c are distinct odd primes. Assume further that $\frac{a^2+a+1}{3}$ is prime and that $\frac{a^2+a+1}{3} | (c^2 + c + 1)$. Then either $a^2 + a + 1 < \frac{3}{4}c$ or $c = \frac{a^2+a+1}{3} - (a + 1)$.*

Proof. Suppose that a and c are distinct odd primes. Assume further that $\frac{a^2+a+1}{3}$ is prime and that $\frac{a^2+a+1}{3}|(c^2+c+1)$. Set $p = \frac{a^2+a+1}{3}$. Consider possible t such that $t^2 + t + 1 \equiv 0 \pmod{p}$. Mod p, there are two residue classes which are solutions, t_1 and t_2. If t_1 is the smaller solution, then they are related by $t_2 = p - t_1 - 1$. Thus, we must have $t_2 > \frac{p}{2}$. We cannot have $a \equiv t_2$ because then we would have

$$a^2 + a + 1 > \left(\frac{p}{2}\right)^2 = \left(\frac{a^2 + a + 1}{3}\right)^2 > \frac{a^4}{9}.$$

This is a contradiction since we must have $a \geq 7$, and so $a^2+a+1 > \frac{a^4}{9}$ is impossible. Thus, a must be the smallest positive integer such that $t^2 + t + 1 \equiv 0 \pmod{p}$.

Now, consider two cases, where $c \equiv a \pmod{p}$ or where $c \not\equiv a \pmod{p}$. In the first case, $c \equiv a \pmod{p}$, we cannot have $c = a$ by assumption, and $a + p$ is even, so we cannot have $c = a + p$. The next option is $c = 2p + a$ and thus we must have $c \geq 2p + a$. but $p \equiv 1 \pmod 3$ and $a \equiv 1 \pmod 3$, so $2p + a \equiv 0 \pmod 3$. Thus, this is not a valid option for c either. Our next possibility is then that $c = 3p + a$, which is even, and so we then have $c \geq 4p + a$, which implies the desired inequality.

Now, consider if c is in the other residue class. This means that $c \equiv a^2 \equiv -(a+1) \pmod{p}$. Consider the simplest case, $c = p - a - 1$. Then

$$c^2 + c + 1 = (p - (a + 1))^2 + p - (a + 1) + 1 = \frac{a^2 + a + 1}{3} - (a + 1),$$

as required by the lemma.

If $c \neq p - a - 1$, then we may, by the same sort of logic as earlier, rule out other small values of c. We can rule out $c = 2p - a - 1$ since this number is even. The next case is

$$c = 3p - a - 1 = 3\left(\frac{a^2 + a + 1}{3}\right) - (a + 1) = a^2 - 1 = (a - 1)(a + 1),$$

which is impossible since c is prime. After that, our next possibility is $c = 4p-a-1$ which is even, so we must have $c \geq 5p-a-1$ which implies that $c > 4(a^2+a+1)$. $\square$

Lemma 35. *There are no solutions to $x^2+x+1 = 3(a^2+a+1)$ where a, x and a^2+a+1 are all odd primes.*

Proof. Assume that $x^2+x+1 = 3(a^2+a+1)$ where a, x and a^2+a+1 are all odd primes. By Lemma 32, we have $a^2+a+1 < \frac{x}{2}$. Thus, $x > 2(a^2+a+1)$, and hence

$$3(a^2+a+1) = x^2+x+1 > x^2 > 4(a^2+a+1),$$

which is a contradiction. $\square$

Lemma 36. *There are no solutions to $x^2+x+1 = (a^2+a+1)(b^2+b+1)$ where x, a, b, a^2+a+1, and b^2+b+1 are all odd primes.*

Proof. The proof is essentially identical to that for Lemma 33. $\square$

Lemma 37. *There are no solutions to $x^2+x+1 = (a^2+a+1)(\frac{b^2+b+1}{3})$ where x, a, b, a^2+a+1 and $\frac{b^2+b+1}{3}$ are all odd primes.*

Proof. Assume that $x^2+x+1 = (a^2+a+1)(\frac{b^2+b+1}{3})$ where x, a, b, a^2+a+1 and $\frac{b^2+b+1}{3}$ are all odd primes. Then

$$3(x^2+x+1) = (a^2+a+1)(b^2+b+1).$$

Since a^2+a+1 is a prime greater than 3 we have $(a^2+a+1)|(x^2+x+1)$. Note that we also have that $(3, x^2+x+1) = 1$ and hence we may apply the second inequality from Lemma 32, to conclude that

$$a^2+a+1 < \frac{2}{9}x. \tag{25}$$

We may also apply Lemma 34, to get either $x = \frac{b^2+b+1}{3} - (b+1)$ or $b^2+b+1 < \frac{3}{4}x$. The second of these would immediately lead to a contradiction with Inequality (25), so we may assume that $x = \frac{b^2+b+1}{3} - (b+1)$. Therefore,

$$x^2+x+1 = \left(\frac{b^2-5b+7}{3}\right)\left(\frac{b^2+b+1}{3}\right).$$

We then must have $\frac{b^2-5b+7}{3} = a^2+a+1$, and that forces that

$$x = \frac{b^2+b+1}{3} - (b+1) = \frac{b^2-5b+7}{3} + b - 3 = a^2+a+1+b-3.$$

This contradicts Inequality (25), and so we have our final contradiction. $\square$

Lemma 38. *There are no solutions to* $x^2+x+1=(\frac{a^2+a+1}{3})(\frac{b^2+b+1}{3})$ *where* x, a, b, $\frac{a^2+a+1}{3}$ *and* $\frac{b^2+b+1}{3}$ *are all odd primes.*

Proof. Assume we have $x^2+x+1=(\frac{a^2+a+1}{3})(\frac{b^2+b+1}{3})$ where x, a, b, $\frac{a^2+a+1}{3}$ and $\frac{b^2+b+1}{3}$ are all odd primes. Note that we cannot have $a=b$ since x^2+x+1 cannot be a perfect square.

We now invoke Lemma 34. We have four cases depending on which of the prongs of Lemma 34 is active for a and b. In Case I, we have $a^2+a+1<\frac{3}{4}x$ and $b^2+b+1<\frac{3}{4}x$. In Case II, we have we have $x=\frac{a^2+a+1}{3}-(a+1)$ and $b^2+b+1<\frac{3}{4}x$. In Case III, we have $a^2+a+1<\frac{3}{4}x$ and $x=\frac{b^2+b+1}{3}-(b+1)$. Finally, in Case IV, we have $x=\frac{a^2+a+1}{3}-(a+1)$ and $x=\frac{b^2+b+1}{3}-(b+1)$.

`Case I.` We obtain a contradiction since we have

$$x^2+x+1=\left(\frac{a^2+a+1}{3}\right)\left(\frac{b^2+b+1}{3}\right)<\left(\frac{x}{4}\right).$$

Cases II and III are essentially identical so we will only discuss Case II. We have $x=\frac{a^2+a+1}{3}-(a+1)$ and $b^2+b+1<\frac{3}{4}x$. Substituting in our equation for x in terms of a we obtain,

$$x^2+x+1=\left(\frac{a^2-5a+7}{3}\right)\left(\frac{a^2+a+1}{3}\right).$$

We then have that $b^2+b+1=a^2-5a+7$. We can rewrite this equation to get that

$$(2-a-b)(a-b-3)=0,$$

but that is impossible to satisfy since a and b are both odd primes.

We then finally have Case IV, where $x=\frac{a^2+a+1}{3}-(a+1)$ and $x=\frac{b^2+b+1}{3}-(b+1)$. We thus have

$$\frac{a^2+a+1}{3}-(a+1)=\frac{b^2+b+1}{3}-(b+1)$$

which implies that either $a=b$ (which we have already ruled out), or that $a+b=2$ which is impossible for primes a and b. $\square$

3. 3 Divides N

We will in this section set f_3 to be the number such that $3^{f_3}||N$. We define S and T by

$$S=\prod_{p||m,p\neq 3} p$$

and
$$T = \prod_{p^2|m, p\neq 3} p.$$
We will set $S = S_1S_2S_3S_4$ where a prime p appears in S_i for $1 \leq i \leq 3$ if $\sigma(p^2)$ is a product of i primes; S_4 will contain all the primes of S where $\sigma(p^2)$ has at least 4 prime factors. We will write $s = \omega(S)$ and write $t = \omega(T)$. We define s_1, s_2, s_3, and s_4 similarly. We will write $S_{i,j}$ to be the primes from S_i which are congruent j modulo 3. In a similar way to use lowercase letters to denote the number of primes in each term as before and in general will use a lowercase letter to denote the number of distinct primes dividing an upper case letter. For example, we set $s_{i,j} = \omega(S_{i,j})$ and will note that $s_{1,1} = 0$. Thus, we do not need to concern ourselves with this split for S_1 since all primes in S_1 are congruent to 2 modulo 3, there is no need to split S_1 further. We will abuse notation slightly and will treat capital letters as both products of distinct primes and as sets containing those distinct primes. Thus, we may also think of lowercase letters as denoting the number of elements in the set formed by an upper case letter.

We have the special exponent is at least 1. That is,
$$1 \leq e \tag{26}$$

We have the following straightforward equations from breaking down the definitions of s_1, s_2, s_3 and s_4.
$$s = s_1 + s_2 + s_3 + s_4. \tag{27}$$
Similarly, we have
$$s_2 = s_{2,1} + s_{2,2}, \tag{28}$$
$$s_3 = s_{3,1} + s_{3,2}, \tag{29}$$
and
$$s_4 = s_{4,1} + s_{4,2}. \tag{30}$$
We define f_4 as the number of prime divisors (counting multiplicity) in N which are not the special prime and are raised to at least the fourth power. From simple counting we obtain
$$e + f_3 + 2s + f_4 \leq \Omega. \tag{31}$$
Due to Lemma 33, any element of $S_{3,1}$ must contribute at least one non-3 prime which is not from S_1. Motivated by Lemma 33, we will define $S_{3,1,T}$ to be the set of elements of $S_{3,1}$ which contribute two prime factors in T or e, and define $S_{3,1,S}$, as those which contribute two prime factors in S. We will similarly define $S_{3,1,ST}$ as those which contribute one to S and one which goes to T or e. We will similarly

define $S_{1,T}$ as the set of elements of S_1 which contribute to T and define $S_{1,S}$ as the set of elements of S_1 which contribute to S. Define $S_{1,e}$ as the set of elements of S_1 which contribute the special prime. We will correspondingly define $s_{3,1,T}$, $s_{3,1,S}$, $s_{3,1,ST}$ $s_{1,T}$, $s_{1,S}$, and $s_{1,e}$. We of course have have $s_{1,e} \leq 1$ but we will not need this here. We then have

$$s_{3,1} \leq s_{3,1,T} + s_{3,1,S} + s_{3,1,ST}. \tag{32}$$

We then define $S_{3,1,\bar{S}T}$ as the set of elements of $S_{3,1,ST}$ which have their S contributing term not arising from an S_1. Similarly define $S_{3,1,S\bar{T}}$ as the set of elements of $S_{3,1,ST}$ which have their T term not arising from an S_1. We define as usual their lowercase variables for counting the number of elements in each set. From Lemma 33, we have that

$$S_{3,1,ST} = S_{3,1,\bar{S}T} \cup S_{3,1,S\bar{T}}.$$

Thus, we have

$$s_{3,1,ST} \leq s_{3,1,\bar{S}T} + s_{3,1,S\bar{T}}. \tag{33}$$

We also have

$$s_1 \leq s_{1,T} + s_{1,S} + s_{1,e}. \tag{34}$$

Lemma 39. *We have*

$$s_1 + s_{2,2} \leq t + s_{2,1} + s_{3,1} + s_{4,1} + 1. \tag{35}$$

Proof. The proof of this lemma is essentially the same as that of Lemma 4 from [23]. □

Next we have

$$s_{2,1} + s_{3,1} + s_{4,1} \leq f_3, \tag{36}$$

since if $x \equiv 1 \pmod 3$, then $x^2 + x + 1 \equiv 0 \pmod 3$.

We also have by counting all the 1 (mod 3) primes which are contributed by primes in S

$$s_1 + 2s_{2,2} + 3s_{3,2} + s_{2,1} + 2s_{3,1} + 4s_{3,2} + 3s_{3,1} \leq f_4 + e + 2s_{2,1} + 2s_{3,1} + 2s_{4,1}.$$

This simplifies to

$$s_1 + 2s_{2,2} + 3s_{3,2} + 4s_{4,2} + s_{4,1} \leq f_4 + e + s_{2,1}. \tag{37}$$

And we of course have

$$4t \leq f_4. \tag{38}$$

We may split $S_{2,2}$ into four sets, $S_{2,2,S}$,$S_{2,2,T}$,$S_{2,2,ST}$, and $S_{2,2,e}$. We define $S_{2,2,e}$ as the set of elements of $S_{2,2}$ which contribute to the special prime at least once. We define $S_{2,2,S}$ as the set of elements of $S_{2,2}$ where both contribute to S, $S_{2,2,T}$ as the set of elements of $S_{2,2}$ where both contribute to T, and $S_{2,2,ST}$ as the set of elements of $S_{2,2}$ where one contributes to S and one contributes to T. We define their lowercase variables as usual. We have

$$s_{2,2} \leq s_{2,2,S} + s_{2,2,T} + s_{2,2,ST} + s_{2,2,e}. \tag{39}$$

We may split $S_{2,1}$ in a similar way into and $S_{2,1,S}$, $S_{2,1,T}$, and $S_{2,1,e}$. We define the lowercase variables as usual. We have

$$s_{2,1} \leq s_{2,1,S} + s_{2,1,T} + s_{2,1,e}. \tag{40}$$

We have

$$s_{1,e} + s_{2,1,e} + 2s_{2,2,e} \leq e. \tag{41}$$

We have

$$s_1 \leq s_{1,S} + s_{1,T} + s_{1,e}. \tag{42}$$

From Lemma 35, Lemma 36, Lemma 37 and Lemma 38, we get that every element of $S_{2,2}$ must contribute at least one prime which does not arise from either an element of S_1 or an element of $S_{2,1}$. Similarly, every element of $S_{2,2,ST}$ must have either a contribution to T or e which does arise from an $S_{2,1}$ or S_1 element or must have a contribution to S which does not arise from an $S_{2,1}$ or an S_1 element. We will set $S_{2,2,S*T}$ as set of those elements which contribute an S element of this form, and $S_{2,2,ST*}$ as the set of those elements which contribute a T term. We define the lower-case counting variables as usual. We then have

$$S_{2,2,ST} \leq S_{2,2,S*T} + S_{2,2,ST*}. \tag{43}$$

We also have

$$2s_{1,S} + 2s_{2,1,S} + s_{2,2,S} + S_{2,2,S*T} + s_{3,1,S} + s_{3,1,\bar{S}T} \leq 2s_{2,1} + 2s_{3,1} + 2s_{4,1} \tag{44}$$

and

$$4s_{1,T} + 4s_{2,1,T} + s_{2,2,T} + s_{2,2,ST*} + s_{3,1,T} + s_{3,1,S\bar{T}} \leq f_4 + e. \tag{45}$$

We have from counting the primes from S which are contributed by S

$$s_{1,S} + s_{2,1,S} + 2s_{2,2,S} + 2s_{3,1,S} + s_{3,1,ST} \leq 2s_{2,1} + 2s_{3,1} + 2s_{4,1}. \tag{46}$$

We have that

$$s + t + 2 = \omega. \tag{47}$$

Note that the $+2$ in Equation (47) arises from 3 and the special prime.

To prove the result we add up our inequalities as follows. We take

$$\frac{9}{25}(\mathbf{26})+\frac{16}{25}(\mathbf{27})+\frac{16}{25}(\mathbf{28})+\frac{16}{25}(\mathbf{29})+\frac{16}{25}(\mathbf{30})+1(\mathbf{31})+\frac{2}{25}(\mathbf{32})+\frac{1}{25}(\mathbf{33})+\frac{8}{25}(\mathbf{34})$$
$$+\frac{2}{25}(\mathbf{35})+1(\mathbf{36})+\frac{6}{25}(\mathbf{37})+\frac{17}{25}(\mathbf{38})+\frac{2}{25}(\mathbf{39})+\frac{8}{25}(\mathbf{40})+\frac{8}{25}(\mathbf{41})+\frac{1}{25}(\mathbf{43})+\frac{7}{50}(\mathbf{44})$$
$$+\frac{2}{25}(\mathbf{45})+\frac{1}{25}(\mathbf{46})+\frac{66}{25}(\mathbf{47}),$$

which yields the desired inequality.

4. 3 Does Not Divide N

For simplicity, we will prove the slightly weaker bound that

$$\frac{302}{113}\omega - \frac{641}{113} \leq \Omega \tag{48}$$

and then discuss the changes needed to improve the constant term. We set $m = 5^{\frac{f_5}{2}}11^{\frac{f_{11}}{2}}S^2T^4U'$. Here we have

$$S = \prod_{p, p||m, p\notin\{5,11\}} p, T = \prod_{p,p^2||m,p\notin\{5,11\}} p.$$

We have $U' = \frac{m}{5^{\frac{f_5}{2}}11^{\frac{f_{11}}{2}}S^2T^4}$ and $U = rad(U')$. That is,

$$U = \prod_{p,p^3|m,p\notin\{5,11\}} p.$$

In other words, S contains the prime divisors other than 5 and 11 which are raised to exactly the second power in the factorization of N. Similarly, T contains the prime divisors other than 5 and 11 which are raised to exactly the fourth power in the factorization of N. Finally, U contains the prime divisors which are raised to at least the sixth power in the factorization of N and are not 5, and 11, and the special prime. We set $s = \omega(S)$, $t = \omega(T)$ and $u = \omega(U)$.

We then have

$$\omega \leq s + t + u - 3. \tag{49}$$

We similarly define f_6 as the set of primes (counting multiplicity) that appear to at least the 6th power. We then have

$$6u \leq f_6 \tag{50}$$

and

$$e + f_5 + 2s + 4t + f_6 + f_{11} \leq \Omega. \tag{51}$$

We do not have an equality in Equation (50) because there may be an element of u raised to a power higher than 6. We will define S_T to be the set of elements of S which arise from T, that is,

$$S_T = \prod_{p|S, p|\sigma(T^4)} p.$$

Similarly, we will define T_S to be the set of elements of T which arise from S:

$$T_S = \prod_{p|T, p|\sigma(S^2)} p.$$

We note that any prime in S_T must be congruent to 1 modulo 5 or must be 5 itself. We will define S_M as the primes in S which are congruent to 1 modulo 5, and set $s_{M5} = \omega(S_{M5})$. We will set $S = S_1S_2S_3S_4S_5$ similarly to how we did in the case when $3 \mid N$, with $p|S_5$ if $\sigma(p^2)$ has 5 or more not necessarily distinct prime factors, and similarly define s_1, s_2, s_3, s_4 and s_5. We then define $S_{M1}, S_{M2} \cdots S_{M5}$ as the intersections of the corresponding S_i and S_M, and then define $s_{M1}, s_{M2} \cdots s_{M5}$ accordingly. We have

$$s \leq s_1 + s_2 + s_3 + s_4 + s_5. \tag{52}$$

We have

$$s_M \leq s_{M1} + s_{M2} + s_{M3} + s_{M4} + s_{M5}. \tag{53}$$

We have that each of the s_i is at least s_{Mi} and thus we have

$$s_{M2} \leq s_2, \tag{54}$$

$$s_{M3} \leq s_3, \tag{55}$$

$$s_{M1} \leq s_4, \tag{56}$$

and

$$s_{M5} \leq s_5. \tag{57}$$

Define T_S to be the set of elements of T which are contributed by S, and define t_S as the lowercase variable as usual. Note that every element of T_S is congruent to 1 modulo 3. We note that

$$t_S \leq t. \tag{58}$$

We note that any prime factor contributed by S cannot itself be in S. To see why, note that any prime p contributed by S is congruent to 1 modulo 3, and in that case $3|\sigma(p^2)$. This contradicts the assumption in this section that $3 \nmid N$. We have from Lemma 31, that every element of S_{3M} must either have all contributions be terms which do not arise from an S_{1M} or must contribute a term which does not arise from an S_1 at all. We will set S_{3MA} as the set of elements of S_{3N} which contribute all terms not arising from S_{1M}. We will set S_{3MB} as the set of elements of S_{3M}

which do not contribute at least one term which does not arise from S_1. We define the lowercase counting variables as usual. We then have

$$s_{3M} \le s_{3MA} + s_{3MB}. \tag{59}$$

We then have

$$4s_{1,m} + 3s_{3MA} + s_{3MB} + s_2 - 3 \le 4t_S + u_S + e_S. \tag{60}$$

Similarly we have

$$4s_1 + s_{3MB} + s_2 - 3 \le 4t_S + u_S + s_e. \tag{61}$$

We also have by the same logic as Lemma 4 from [23] that

$$s_1 + s_2 \le t_S + u_S + 1. \tag{62}$$

We also have from counting all the various s_i contributions that

$$s_1 + 2s_2 + 3s_3 + 4s_4 + 5s_5 \le 4t_S + u_S + e_S. \tag{63}$$

We now turn to the equations which allow us to bound the number of primes in S_M. To do this we need lower bounds on the contribution to S from elements in T. We define T_{S1}, T_{S2}, T_{S3}, T_{S4} and T_{S5} as follows: for $1 \le i \le 4$ we define T_{Si} to be the set of elements of T_S which contribute exactly i primes, and we define T_{S5} to be the set of those elements of T_S which contribute at least 5 primes. We define T_M to be the set of elements of T which are congruent to 1 modulo 5. Note that

$$t_S \le t_{S1} + t_{S2} + t_{S3} + t_{S4} + t_{S5}. \tag{64}$$

We have

$$t_{S1} + 2t_{S2} + 3t_{S3} + 4t_{S4} + 5t_{S5} \le 2s_M + 4t_M + u_T + e_T + f_5 + f_{11} \tag{65}$$

where u_t and e_t are defined analogously to u_s and e_s.

We note that every element in T_S is congruent to 1 modulo 3, and that if $x \equiv 1 \pmod 3$, we have that $x^4 + x^3 + x^2 + x + 1 \equiv 2 \pmod 3$. But every element contributed by a prime in S must be congruent to 1 modulo 3. So if $p \in T_S$ then $\sigma(p^4) \equiv 2 \pmod 3$. Thus, any element of $T_{i,S}$ when i is odd must contribute at least one prime which is congruent to 1 modulo 3 (and hence contribute a prime not in S), since a product of an even number of numbers all congruent to 2 modulo 3 will be congruent to 1 modulo 3. Thus we have,

$$t_{S2} + t_{S4} \le 4t_M + u_t + e_t. \tag{66}$$

The next set of inequalities seeks to deal with the problem that we may have very large T_{S1}. From Lemma 6, it follows that no element of T_{S1} can give rise to an element of S_1. For i with $2 \leq i \leq 5$ Define S_{i*} as the elements of S_i arising from T_{1S}. We define the lowercase counting variables as usual. We then have

$$t_{S1} \leq s_{2*} + s_{3*} + s_{4*} + s_{5*} + t_M + u_t + e_t. \tag{67}$$

We have

$$s_{4*} \leq s_4, \tag{68}$$

and

$$s_{5*} \leq s_5. \tag{69}$$

We have from Lemma 4 and Lemma 5, that

$$s_1 + s_{3*} - 1 \leq t_S + u. \tag{70}$$

We have from counting our contribution to the special prime that

$$e_S + e_T \leq e. \tag{71}$$

We also have

$$u_S + u_T \leq f_6. \tag{72}$$

We note that since every element in T_M is 1 modulo 5, we have

$$t_m \leq f_5. \tag{73}$$

We also have by Lemma 6, that any contribution from S_{2*} which contributes both terms to T must also contribute to f_{11}. We set S_{2*T} to be the set of those elements of S_2* which contribute both terms to T, and we set S_{2*UE} to be those which contribute at least one term to either u or e. We then have,

$$s_{2*} \leq s_{2*T} + s_{2*UE}, \tag{74}$$

$$s_{2*T} \leq f_{11}, \tag{75}$$

and

$$s_{2*UE} \leq u - 1. \tag{76}$$

We have as before that the special prime must be raised to at least the first power and thus,

$$1 \leq e. \tag{77}$$

Finally we have that

$$\omega \leq s + t + u + 3. \tag{78}$$

The 3 comes from the special prime, the possibility of division by 5 and the possibility of division by 11.

To prove Inequality (48) we then add our inequalities as follows. We take

$$\frac{302}{113}(\mathbf{49}) + \frac{53}{113}(\mathbf{50}) + 1(\mathbf{51}) + \frac{76}{113}(\mathbf{52}) + \frac{8}{113}(\mathbf{53}) + \frac{8}{113}(\mathbf{54}) + \frac{2}{113}(\mathbf{55})$$
$$+\frac{8}{113}(\mathbf{56}) + \frac{8}{113}(\mathbf{57}) + \frac{150}{113}(\mathbf{58}) + \frac{6}{113}(\mathbf{59}) + \frac{2}{113}(\mathbf{60}) + \frac{4}{113}(\mathbf{61}) + \frac{26}{113}(\mathbf{62})$$
$$+\frac{26}{113}(\mathbf{63}) + \frac{12}{113}(\mathbf{64}) + \frac{4}{113}(\mathbf{65}) + \frac{4}{113}(\mathbf{66})\frac{8}{113}(\mathbf{67}) + \frac{8}{113}(\mathbf{68}) + \frac{8}{113}(\mathbf{69})$$
$$+\frac{8}{113}(\mathbf{70}) + \frac{32}{113}(\mathbf{71})\frac{58}{113}(\mathbf{72}) + \frac{40}{113}(\mathbf{73}) + \frac{8}{113}(\mathbf{74}) + \frac{8}{113}(\mathbf{75}) + \frac{8}{113}(\mathbf{76})$$
$$+\frac{81}{113}(\mathbf{77}).$$

It follows from the results in [4] that if one has any of $5^2||N$, $5^4||N$, $11^2||N$, or $11^4||N$ then one must have at least one prime which is not the special prime raised to at least the 6th power, and hence have $u \geq 1$. We thus may adjust the above equations slightly: If $(N, 55) = 1$, we have instead of Equation (49), we have $\omega = s + t + u - 1$ and $f_5 = f_{11} = 0$. In this case we get a bound of

$$\frac{302}{113}\omega - \frac{286}{113} \leq \Omega. \tag{79}$$

Alternatively, one must have $5^6|N$ or $11^6|N$ if either $5|N$ or $11|N$. We can without too much work check that all of these force a bound at least as tight as Inequality (79). Thus we always have that bound.

5. Improved Norton Type Results

Norton [14] proved two types of results. First, he proved lower bounds for $\omega(N)$ in terms of the smallest prime factor of N. Second, he proved lower bounds for N in terms of its smallest prime factor. In this section, we will slightly improve Norton's first type of result and show how we can combine that with the Ochem-Rao type results to substantially improve the second type of result. Set P_n to be the nth prime number. For $n > 1$ Norton defined $a(n)$ as the integer such that

$$\prod_{r=n}^{n+a(n)-2} \frac{P_r}{P_r - 1} < 2 < \prod_{r=n}^{n+a(n)-1} \frac{P_r}{P_r - 1}. \tag{80}$$

It is easy to see that if N is an odd perfect number with smallest prime divisor P_n, then N must have at least $a(n)$ distinct prime divisors. In fact, although Norton does not state this explicitly, this statement also applies to any odd abundant number N. The function $a(n)$ is also closely related to Mertens theorem which relies on the same product. Thus, study of $a(n)$ is a natural object even if one is not strongly interested in odd perfect numbers. Norton proved

Theorem 3. ([14]). *Let N be an odd perfect number with smallest prime divisor P_n, largest prime divisor P_s, and let b be a constant less than $\frac{4}{7}$. Then we have:*

1.
$$a(n) = Li(P_n^2) + O\left(n^2 e^{-\log^b n}\right). \tag{81}$$

2.
$$a(n) = \frac{1}{2}n^2 \log n + \frac{1}{2}n^2 \log\log n - \frac{3}{4}n^2 + \frac{n^2 \log\log n}{2\log n} + O\left(\frac{n^2}{\log n}\right). \tag{82}$$

3.
$$P_s \geq P_{n+a(n)-1} = P_n^2 + O\left(n^2 e^{-\log^b n}\right). \tag{83}$$

4.
$$P_s \geq P_{n+a(n)-1} = n^2 \log^2 n + 2n^2 \log n \log\log n - 2n^2 \log n + n^2(\log\log)^2 + O(n^2). \tag{84}$$

5.
$$\log N > 2P_n^2 + O\left(n^2 e^{-\log^b n}\right). \tag{85}$$

Note that only Equation (85) is using non-trivial material about odd perfect numbers. The bounds for P_s apply to any odd abundant number, and the bounds for $a(n)$ do not depend on odd perfect numbers at all. These bounds of Norton are not by themselves constructive; Norton proved slightly weaker constructive bounds.[4].

Theorem 4. ([14]). *Let N be an odd perfect number with smallest prime divisor P_n, and set $P_s = P_{a(n)-n-1}$. Then we have*

1.
$$a(n) > n^2 - 2n - \frac{n+1}{\log n} - \frac{5}{4} - \frac{1}{2n} - \frac{1}{4n\log n}. \tag{86}$$

2. *As long as $n \geq 9$,*
$$\log N > 2P_s\left(1 - \frac{1}{2\log P_s}\right) - 2P_n\left(1 + \frac{1}{2\log P_n}\right) + 6\log P_n + 2\log P_{n+1} - \log P_s. \tag{87}$$

These bounds are explicit with the cost of being weaker in form than the bounds in Theorem 3. To prove Equation (86), Norton did have to use results about odd perfect numbers. In fact, Norton's method uses some early results ruling out specific

[4]Prior to Norton a similar non-constructive but more general bound was proven which gives a lower bound for the $\omega(n)$ in terms of α and p where n satisfies $\frac{\sigma(n)}{n} \geq \alpha$ and n has smallest prime factor p [20].

forms of odd perfect numbers, although the forms ruled out only allow improvement in the lower order terms of his inequality. Subsequent results in this section can be thought of as using similar ideas. Because we have stronger results on what an odd perfect number can look like, we can actually improve the constant in front of the lead terms. It is also worth noting that for large values of n, Norton's explicit bound for $a(n)$ gives a better result than the often cited bound of Grun [7], which shows that if N is an odd perfect number with smallest prime divisor p, then

$$\frac{3}{2}p - 2 \leq \omega(N). \tag{88}$$

Norton's result focus on $a(n)$, but for some purposes it is more natural to look at the function $b(p)$, defined for odd primes p, where $b(p) = a(n)$ where $p = P_n$. We will examine the behavior of both functions in this section. While Norton's results in work for abundant or perfect numbers, we will also be interested in how they can be strengthened for odd perfect numbers. In this context, we will define $b_o(p)$ to be the minimum of the number of distinct prime divisors of any odd perfect number with smallest prime divisor p. We will set $b_o(p) = \infty$ when there are no odd perfect numbers with smallest prime divisor p. We will define $a_o(n)$ in analogous fashion. Trivially one has $b_o(p) \geq b(p)$ and $a_o(n) \geq a(n)$. One would like to be able to prove that $b_o(p) > b(p)$ but this seems very difficult.

In this section, we will prove strengthened versions of Norton's constructive results bounding $a(n)$ from below and a similar one for $b(p)$ although they will still fall slightly short of the non-constructive results. In the next section, we will use these results to construct a general framework to use Ochem and Rao type results to get explicit inequalities similar to Inequality (87) which in general are better than Norton (both his explicit form and his non-constructive form in Equation (85)). We will then use this framework and our earlier results to construct a strong lower bound for the size of an odd perfect number in terms of its smallest prime factor. We will write $S(x) = \prod_{p \leq x} \frac{p}{p-1}$. We will write $\vartheta(x)$ to be Chebyshev's second function, that is,

$$\vartheta(x) = \sum_{p \leq x} \log p.$$

We need the following lemma.

Lemma 40. *For any prime $p > 2$, we have $b(p) \geq p$.*

Proof. Assume that $b(p) = m$. Then since $\frac{x}{x-1}$ is a decreasing function for positive x and the ith prime after p is at least $p + i$, we must have

$$2 < \left(\frac{p}{p-1}\right)\left(\frac{p+1}{p}\right)\left(\frac{p+2}{p+1}\right)\cdots\left(\frac{p+b(p)-1}{p+b(p)-2}\right) = \frac{p+b(p)-1}{p-1}.$$

Thus

$$2p - 2 < p + b(p) - 1,$$

and so $b(p) > p - 1$, and so $b(p) \geq p$. □

Note that variants of the above lemma are very old. Often this result is normally stated simply that an odd perfect number must have more distinct prime factors than its smallest prime divisor. The lemma when stated applying just to perfect numbers seems to date back to Servais [21] and has been proved repeatedly such as in [17]. In that context, it is worth noting that there is also a slightly stronger version in the literature which again has been proven a few times. In this case, the oldest version appears to be due to Grun [7]. As previously discussed, Grun proved that if N is an odd perfect number with least prime divisor p, then $\frac{2}{3}\omega(N) + 2 \geq p$. Again, the proof can, with no substantial effort, be generalized to a statement about $b(p)$. In Grun's proof, the key observation is that odd primes must differ by at least 2, and therefore one can instead use the inequality

$$2 < \left(\frac{p}{p-1}\right)\left(\frac{p+2}{p+1}\right)\left(\frac{p+4}{p+3}\right)\cdots\left(\frac{p+2b(p)-2}{p+b(p)-3}\right),$$

and then estimate the quantity on the right-hand side. As with the lemma of Servais, Grun's lemma applies to any odd perfect number or odd abundant number although it is normally phrased simply for odd perfect numbers.

We will need a few explicit estimates of certain functions of primes. We have from [3] that

$$\frac{x}{\log x}\left(1 + \frac{0.992}{\log x}\right) \leq \pi(x) \leq \frac{x}{\log x}\left(1 + \frac{1.2762}{\log x}\right), \tag{89}$$

with the lower bound valid if $x \geq 599$ and the upper bound valid for all $x > 1$. We also have for $n \geq 2$,

$$P_n \geq n\left(\log n + \log\log n - 1 + \frac{32}{31(\log n)^2}\right). \tag{90}$$

The above bound for P_n follows from the following bound in [3] which differs only in the last term:

$$P_n \geq n\left(\log n + \log\log n - 1 + \frac{\log\log x - \frac{9}{4}}{\log x}\right). \tag{91}$$

To obtain Inequality (90), we note that for if $n \geq 35312$, we have that

$$\frac{\log\log n - 9/4}{\log n} \geq \frac{32}{31(\log n)^2}.$$

We verify the inequality by direct computation for all n with $2 \leq n \leq 35312$. We will write

$$\underline{P}_n = n\left(\log n + \log\log n - 1 + \frac{32}{31(\log n)^2}\right).$$

Lemma 41. *If A and B are real numbers, with $A \geq B \geq 6$ and they satisfy*

$$A + \frac{1}{2A} \geq 2B - \frac{1}{B},$$

then

$$A \geq 2B - \frac{5}{4B} - \frac{1}{37B^2}.$$

Proof. Note that $A + \frac{1}{2A}$ is a positive increasing function in A. Hence, to prove this one simply needs to verify that if $B \geq 6$, then

$$2B - \frac{5}{4B} - \frac{1}{37B^2} + \frac{1}{2(2B - \frac{5}{4B} - \frac{1}{37B^2})} \leq 2B - \frac{1}{B}.$$

$\square$

For $x > 1$ we have [19]

$$e^{\gamma}(\log x)\left(1 - \frac{1}{2\log^2 x}\right) < S(x) < e^{\gamma}(\log x)\left(1 + \frac{1}{\log^2 x}\right) \tag{92}$$

where γ is Euler's constant. We will write

$$Q(x, y) = \prod_{y<p\leq x} \frac{p}{p-1} = \frac{S(x)}{S(y)}$$

and will assume that $x > y$. We have from Equation (92) that

$$\frac{\log x}{\log y}\left(\frac{1 - \frac{1}{2\log^2 x}}{1 + \frac{1}{\log^2 y}}\right) < Q(x, y) < \frac{\log x}{\log y}\left(\frac{1 + \frac{1}{\log^2 x}}{1 - \frac{1}{2\log^2 y}}\right). \tag{93}$$

Theorem 5. *For all $n > 1$ we have*

$$a(n) \geq \frac{n^2}{2}\left(\log n - \frac{3}{2}\log\log n + \frac{1}{20} + \frac{\log\log n}{\log n}\right) - n + 1.$$

Proof. We may verify from direct computation that the above inequality is valid for $n \leq 117$, and so we may assume that $n \geq 118$ or equivalently that $p \geq 647$. Our plan to estimate $a(n)$ is to set $y = P_n$ in Equation (93), find a lower bound on x such that $Q(x, y) > 2$, and then estimate $\pi(x)$ since we will have $a(n) = \pi(x) - n + 1$. We have from Equation (93) that

$$2 < \frac{\log x}{\log y}\left(\frac{1 + \frac{1}{\log^2 x}}{1 - \frac{1}{2\log^2 y}}\right)$$

which implies that

$$\log x + \frac{1}{2\log x} \geq 2\log y - \frac{1}{\log y}.$$

We then have from Lemma 41 that

$$\log x \geq 2\log y - \frac{5}{4\log y} - \frac{1}{37(\log y)^2}. \quad (94)$$

(In the above use of the Lemma we are setting $A = \log x$ and $B = \log y$.) Note that getting a lower bound $a(n)$ is exactly the same as lower bounding the minimum x, such that $Q(x, p_n) > 2$, then applying lower bound on $\pi(x) - \pi(y) = \pi(x) - n + 1$. It is not hard to see that Equation (94) yields that as long as $y \geq 541$ that

$$x \geq y^2\left(1 - \frac{5}{4\log y} + \frac{1}{2(\log y)^2}\right). \quad (95)$$

We now need to estimate $a(n)$ by estimating $\pi(x) - n + 1$. We then have from Equation (95) and Equation (90), along with the fact that the function $j(s) = 1 - \frac{5}{4s} + \frac{1}{2s^2}$ is increasing for $s \geq 1/10$ that

$$a(n) \geq \pi\left((\underline{P}_n)^2\left(1 - \frac{5}{4\log \underline{P}_n} + \frac{1}{2(\log \underline{P}_n)^2}\right)\right) - n - 1 \quad (96)$$

We have the trivial estimate that $\underline{P}_n \geq n$. Way again apply that $f(s)$ is increasing in s, and substitute in the definition of $\underline{P}_n$, and set $t = \log n$ to obtain

$$a(n) \geq \pi\left(n^2\left(t + \log t - 1 + \frac{32}{31t^2}\right)^2\left(1 - \frac{5}{4t} + \frac{1}{2t^2}\right)\right) - n - 1. \quad (97)$$

When $t > 4.77$, one has that

$$\left(t + \log t - 1 + \frac{32}{31t^2}\right)^2\left(1 - \frac{5}{4t} + \frac{1}{4t^2}\right) \geq t^2 - \frac{\log t}{2}.$$

So for n in the range under discussion we have

$$a(n) \geq \pi\left(n^2(t^2 - \frac{\log t}{2})\right) - n + 1 = \pi\left(n^2((\log n)^2 - \frac{\log\log n}{2})\right) - n + 1. \quad (98)$$

We wish to apply Equation (89) to (98). To do so, we need a lower bound on

$$\frac{1}{\log\left(n^2((\log n)^2 - \frac{\log\log n}{2})\right)}.$$

It is not hard to check that as long as $n > e^e$ one has that

$$\frac{1}{\log\left(n^2((\log n)^2 - \frac{\log\log n}{2})\right)} \geq \frac{1}{2\log n}\left(1 - \frac{\log\log n}{\log n}\right). \quad (99)$$

We can now apply Equation (99) to Equation (98) and (89) to get that

$$a(n) \geq \frac{n^2\left(t^2 - \frac{t\log t}{2}\right)\left(1 - \frac{\log t}{t}\right)}{2t}\left(1 + 0.992\left(\frac{1}{2t} - \frac{\log t}{2t^2}\right)\right) - n - 1. \tag{100}$$

We have again in the above for convenience written $\log n$ as t. A little work then shows that for $n \geq 43$, we have that the right-hand side of Equation (100) is at least

$$\frac{n^2}{2}\left(\log n - \frac{3}{2}\log\log n + \frac{1}{20} + \frac{\log\log n}{\log n}\right) - n + 1$$

which proves the theorem. $\square$

Note that the bound given by Theorem 5 is tighter than the bound from Equation (86). We similarly have an interest in estimating $b(p)$.

Theorem 6. *For all primes $p > 2$, we have*

$$b(p) \geq \frac{p^2}{2\log p}\left(1 - \frac{0.754}{\log p} - \frac{0.745}{(\log p)^2} - \frac{0.247}{(\log p)^3} + \frac{0.631813}{(\log p)^4}\right) - \pi(x) + 1. \tag{101}$$

and

$$b(p) \geq \frac{p^2}{2\log p}\left(1 - \frac{0.754}{\log p} - \frac{0.745}{(\log p)^2} - \frac{0.247}{(\log p)^3} + \frac{0.631813}{(\log p)^4}\right) - \frac{p}{\log p}(1 + \frac{1.2726}{\log p}) + 1. \tag{102}$$

Proof. We will prove only the second of the two inequalities (the proof for the first statement is nearly identical). The first few steps in this proof are essentially identical to those in the proof of Theorem 5. We again assume that $p \geq 647$, and proceed until we reach Equation (95). And as before we estimate

$$\pi(x) - n + 1.$$

We need to lower bound the left-hand side of

$$b(p) \geq \pi\left(p^2\left(1 - \frac{5}{4\log p} + \frac{1}{2(\log p)^2}\right)\right) - \pi(p) + 1. \tag{103}$$

We need to apply Inequality (89). We note that although the lower bound on Equation (89) requires that the argument of x be at least 599, we have that in this case since $p \geq 647$. We also need a lower bound estimate for

$$\frac{1}{\log(p^2(1 - \frac{5}{4\log p} + \frac{1}{2(\log p)^2}))}.$$

It is not hard to verify that when p in our range we have

$$\frac{1}{\log(p^2(1-\frac{5}{4\log p}+\frac{1}{2(\log p)^2})} \geq \frac{1}{2\log p}\left(1-\frac{5}{8(\log p)^2}-\frac{21}{32(\log p)^3}\right). \tag{104}$$

We will set $t = \log p$, and then use Equation (104) to apply Equation (89) to (103) to get that

$$b(p) \geq \frac{p^2}{2t}\left(1-\frac{5}{4t}+\frac{1}{2t^2}\right)\left(1-\frac{5}{8t^2}-\frac{21}{32t^3}\right)\left(1+\frac{0.992}{2t}\left(1-\frac{5}{8t^2}-\frac{21}{32t^3}\right)\right) - \frac{p}{t}(1+\frac{1.2726}{t})+1. \tag{105}$$

We need then to estimate

$$I_2(t) = \left(1-\frac{5}{4t}+\frac{1}{2t^2}\right)\left(1-\frac{5}{8t^2}-\frac{21}{32t^3}\right)\left(1+0.992\left(1-\frac{5}{8t^2}-\frac{21}{32t^3}\right)\right).$$

We have

$$I_2 = 1-\frac{0.754}{t}-\frac{0.745}{t^2}-\frac{0.247}{t^3}+\frac{0.631813}{t^4}+E(t)$$

where

$$E(t) = \frac{0.369375}{t^5}-\frac{0.160813}{t^6}-\frac{0.198109}{t^7}-\frac{0.0635742}{t^8}+\frac{0.106805}{t^9}.$$

We note that $E(t)$ is positive when $t > 1$ which is satisfied in our range. Thus we conclude that

$$I_2(t) \geq 1-\frac{0.754}{t}-\frac{0.745}{t^2}-\frac{0.247}{t^3}+\frac{0.631813}{t^4}$$

which proves the theorem. □

Note that although Theorem 6 and Theorem 5 both give the asymptotically correct values, in practice Theorem 6 is stronger. This is due to Theorem 5 requiring that we also use a lower bound estimate for P_n in terms of n.

While we cannot directly show that $b_o(p) > b(p)$, we can get partial results of this form. In particular, $b(p) = 3$, but $b_o(3) \geq 10$ [13]. Similarly, $b(5) = 7$, and $b_o(5) \geq 12$ [11]. We will also prove a similar result for other small values of $b(p)$ using the fact that the largest prime divisor of an odd perfect number must be at least 10^8 [6].

Proposition 1. *We have $b_o(p) \geq b(p) + 1$, for $p \leq 397$.*

Proof. We will show the calculation for $p = 7$. The calculation is nearly identical for the other primes in question. We note that $b(7) = 15$. Now, assume N is an odd perfect number with smallest prime divisor 7, and with exactly 15 distinct prime divisors. Then we have

$$2 = \frac{\sigma(N)}{N} < H(N) \leq \frac{7}{6}\frac{11}{10}\frac{13}{12}\frac{17}{16}\cdots\frac{53}{52}\frac{59}{58}\frac{10^8+1}{10^8} < 1.994.$$

This is a contradiction. $\square$

This proposition stops at 397 because the relevant product is actually greater than 2 for the next prime, 401. The result could be extended if the result from [6] could be extended further; however, extending that result (say to that an odd perfect number must have a prime divisor which is at least 10^9) would likely take either very heavy new computations or would take some fundamental new insight. That said, it is plausible that a similar result could be proved just for odd perfect numbers not divisible by any prime less than some bound, and this would allow one to extend the above proposition in this case. This proposition also allows us to extend some other results of Norton. For example, it is frequently mentioned that Norton proved that an odd perfect number not divisible by 3, 5 or 7 must have at least 27 distinct prime factors. Proposition 1 allows one to replace 27 in that result with 28.

We also have as a consequence of Theorem 6 the following corollary.

Corollary 3. *If $p \geq 11$, then $b(p) \geq 2p + 2$.*

Using Proposition 1, Corollary 3 and the earlier remarks for $p = 3$ and $p = 5$, we can combine this with Theorem 6 to obtain with a little work a result specifically about odd perfect numbers.

Corollary 4. *Let N be an odd perfect number with smallest prime factor p. Then we have $\omega(N) \geq 2p + 2$.*

Proof. The result is essentially just Corollary 3 except for $p = 3, 5, 7$. We may deal with $p = 3$ by recalling that an odd perfect number must have at least 10 distinct prime factors and $10 \geq 2(3) + 2$. Since an odd perfect number not divisible by 3 must have at least 12 prime factors, 5 is likewise handled. Since $b(7) = 15$, we have that $b_o(7) \geq 16$ by Proposition 1. And so the result is proven. $\square$

Note that Corollary 4 is tighter than Grun's result for all odd primes p. One can see from examples like 945 that this bound really does require that N is an odd perfect number, unlike Grun's bound which applies also to odd abundant numbers.

It is easy to see from the definition of $a(n)$ that $a(n + 1) \geq a(n)$ for all $n \geq 2$. However, Norton's bounds do not appear by themselves to be tight enough to conclude that $a(n + 1) > a(n)$ for all $n \geq 2$. But we can use Corollary 3 to prove this result.

Proposition 2. *For all $n \geq 2$, $a(n+1) \geq a(n) + 1$. Equivalently, if P_n is an odd prime and P_{n+1} is the next prime after P_n, then $b(P_{n+1}) \geq b(P_n) + 1$.*

Proof. We can verify that the statement is true for any prime $p \leq 17$, so we may without loss of generality assume that $P_{n+1} > P_n \geq 19$. Assume that $b(P_n) = b(P_{n+1}) = m$. This means we have that

$$\prod_{r=n}^{n+m-2} \frac{P_r}{P_r - 1} < 2 < \prod_{r=n}^{n+m-1} \frac{P_r}{P_r - 1}. \tag{106}$$

and

$$\prod_{r=n+1}^{n+m-1} \frac{P_r}{P_r - 1} < 2 < \prod_{r=n+1}^{n+m} \frac{P_r}{P_r - 1}. \tag{107}$$

We then have

$$\prod_{r=n+1}^{n+m} \frac{P_r}{P_r - 1} = \left(\prod_{r=n}^{n+m-2} \frac{P_r}{P_r - 1} \right) \left(\frac{P_n - 1}{P_n} \right) \left(\frac{P_{n+m-1}}{P_{n+m-1} - 1} \right) \left(\frac{P_{n+m}}{P_{n+m-1} - 1} \right). \tag{108}$$

However, we have from Equation (106) that the first term on the right-hand side of Equation (108) is less than 2. We claim that the remaining terms are less than 1, which would mean that the right-hand side of Equation (107) would be both greater than 2 and less than 2 which is a contradiction. It just remains to show that

$$\left(\frac{P_n - 1}{P_n} \right) \left(\frac{P_{n+m-1}}{P_{n+m-1} - 1} \right) \left(\frac{P_{n+m}}{P_{n+m-1} - 1} \right) < 1.$$

We note that $b(P_n) \geq 2P_n + 2$, and thus $P_{n+m-1} \geq 2P_n + 1$. We then have $P_{n+m} \geq 2P_n + 3$. Thus we have that

$$\left(\frac{P_n - 1}{P_n} \right) \left(\frac{P_{n+m-1}}{P_{n+m-1} - 1} \right) \left(\frac{P_{n+m}}{P_{n+m-1} - 1} \right) \leq \left(\frac{x-1}{x} \right) \left(\frac{2x+1}{2x} \right) \left(\frac{2x+3}{2x+2} \right),$$

where $P_n = x$. However, we have that

$$\left(\frac{x-1}{x} \right) \left(\frac{2x+1}{2x} \right) \left(\frac{2x+3}{2x+2} \right) = \frac{4x^3 + 4x^2 - 5x - 3}{4x^3 + 4x^2} < 1.$$

This completes the proof. $\square$

In a similar vein, one can easily modify the above proof to obtain a slightly more general result.

Proposition 3. *For any constant c there are only finitely many n where*

$$a(n+1) - a(n) \leq c.$$

We have just shown that $a(n)$ is strictly increasing in n. This is the same as saying that the first difference, $a(n+1)-a(n)$, is always positive. One might naturally wonder about the behavior of the second differences of $a(n)$. Since $a(n)$ asymptotically behaves like $\frac{1}{2}n^2 \log n$ which has positive and indeed slightly increasing second differences, one might hope that $a(n)$ at least has always positive second differences. Alas, this is not the case. Let $f(n)$ be the second difference of $a(n)$, that is,

$$f(n) = a(n+1) + a(n-1) - 2a(n).$$

Generally, $f(n)$ is positive. However, $f(31) = -5$, and $f(100) = -144$. What is going on here? The key issue appears to be that both of these values correspond to primes which occur right after a large gap. We say that a prime P_n occurs after a record setting gap if $P_n - P_{n-1}$ is larger than it is for any other choice of smaller n. In particular, the 30th prime is 113, and then there is a record-setting gap to the 31st prime of 127. Similarly, the 99th prime is 523 and then there is a record setting gap to the 100th prime of 541. This should not be surprising. Because there are unusual gaps here, $a(30)$ and $a(99)$ need to be extra large since the relevant products lack any smallish primes other than P_{30} and P_{99}. (Remember that the smaller a prime the more it contributes to our product.) We can check this intuition by looking at when $f(n) = 1$ and noting the two smallest examples of this occur at $n = 10$, corresponding to the record setting gap between 23 and 29, and at $n = 25$, corresponding to the record setting gap between 83 and 89. Note that we can have $f(n) = 1$ when the gap is not a record setting gap, such as at $n = 35$, which corresponds to the large but not record setting gap between 139 and 149. This discussion leads to four questions about the behavior of $f(n)$ and one about $a(n+1) - a(n)$.

1. Are there infinitely many values of n where $f(n)$ is negative?
2. Is the set of n where $f(n) < 0$ a subset of those n where P_n is after a record setting gap?
3. Are there infinitely many n where P_n occurs at a record setting gaps and $f(n)$ is positive?
4. Does $f(n)$ take on every integer value? In particular, is $f(n)$ ever zero?
5. Does $a(n+1) - a(n)$ take on every positive integer value?

6. Hybrid Bounds

We wish to combine the Norton type results together with the Ochem-Rao type results to get a strong lower bound on the size of an odd perfect number in terms of its smallest prime factor. We will write $b_2(p) = b(p) + \pi(p) - 1$.

Let S be a set of odd primes. We say N is an *S-avoiding OPN* if N is an odd perfect number not divisible by any prime in S. (Here OPN arises from "Òdd Perfect Number.") Notice in particular that if the smallest prime factor of N is p, then N is an S-avoiding OPN with S the set of odd primes strictly less than p. Given S a set of primes (possibly empty), and α, and β to be real numbers, we will write OR(α, β, S) for the statement "For any S-avoiding OPN, we have $\Omega(N) \geq \alpha\omega(N) + \beta$." In this framework, Ochem and Rao's original result of Inequality (1) is the statement OR($\frac{18}{7}$, $\frac{-31}{7}$, $\emptyset$). Similarly, Inequalities (3) and (4) can be stated as OR($\frac{8}{3}$,$\frac{-7}{3}$, $\{3\}$) and OR($\frac{21}{8}$, $\frac{-39}{8}$, $\emptyset$). Theorem 2 can be stated OR($\frac{302}{113}$,$\frac{-286}{113}$,$\{3\}$) and OR($\frac{66}{25}$,-5,$\emptyset$).

Theorem 7. *Let S be a finite set of odd primes. Let α and β be real numbers with $\alpha > 2$. Let M be the maximum of S. Assume that $p > M$. Let N be an odd perfect number with smallest prime factor p, and also satisfying $\alpha\omega(N) + \beta \geq 0$. Set $Q = P_{n+b(p)-1}$. Then we have,*

$$\log N \geq (\log p)\left((\alpha - 2)(b(p)) - \beta + 1\right) + 2(\vartheta(Q) - \vartheta(p)) - \log Q.$$

Proof. Assume as given and note that every prime factor of an odd perfect number except possibly the special prime must be raised to at least the second power. This contributes the $2(\vartheta(P) - \vartheta(p)) - \log Q$ term (where in the worst case scenario Q is the special prime). However, we have an additional contribution of the remaining primes which are forced by our lower bound for $\Omega(N)$. Each of those primes is at least p, and there are at least $((\alpha-2)\omega(N)+\beta+1$ such primes (with the $+1$ coming from our special prime only being raised to the first power rather than the second). This gives us the other term above. ☐

We will need the following result from [19]:

$$x\left(1 - \frac{1}{2\log x}\right) < \vartheta(x) < x\left(1 + \frac{1}{2\log x}\right). \tag{109}$$

Here $\vartheta(x)$ is Chebyshev's second function, that is, $\vartheta(x) = \sum_{p \leq x} \log p$, and the upper bound is valid for $x > 563$ and the lower bound is valid for $x > 1$. We have the following as an immediate corollary of Equation (101).

Corollary 5. *Let N be an odd perfect number with smallest prime factor p. Then we have*

$$b_2(p) \geq \frac{p^2}{2\log p}\left(1 - \frac{0.754}{\log p} - \frac{0.745}{(\log p)^2} - \frac{0.247}{(\log p)^3} + \frac{0.631813}{(\log p)^4}\right). \tag{110}$$

We can use this sort of result to get results stronger than Norton's lower bounds for $\log N$ in Inequality (85) and Inequality (87). We have, using our previous bounds and a little algebra, the following lemma.

Lemma 42. *Let p be an odd prime greater than 3. Set $t = \log p$. Then we have*

$$P_{b_2(p)} \geq p^2 I_3(t) \tag{111}$$

where

$$I_3(t) = 1 - \frac{0.754}{t} - \frac{2.5\log t}{t^2} - \frac{1.808}{t^2} - \frac{0.55\log t}{t^3} + \frac{0.41(\log t)^2}{t^4} + \frac{0.2\log t}{t^4} + \frac{3.6}{t^4}. \tag{112}$$

We can use Lemma 7 and Lemma 42 with our bound for $b(p)$ from Theorem 6 as well as the statement $\mathrm{OR}(\frac{8}{3}, \frac{7}{3}, \{3\})$ and Ochem and Rao's bound that $N > 10^{1500}$ to obtain a new lower bound for an odd perfect number in terms of its smallest prime factor.

Theorem 8. *Let N be an odd perfect number with smallest prime divisor p. Then we have that*

$$\log N \geq p^2 \left(\frac{7}{3} - \frac{2.51}{t} - \frac{2.5\log t}{t^2} - \frac{1.31}{t^2} - \frac{3.2\log t}{t^3} - \frac{4.1\log t}{t^4} \right). \tag{113}$$

Note that we have used $\mathrm{OR}(\frac{8}{3}, \frac{7}{3}, \{3\})$ rather than our new bound since our main theorem is not better until $\omega \geq 34$. One can derive a similar result, using the main theorem of this paper which will be weaker when N is divisible by a small prime p.

7. On the Strength of Restrictions About an Odd Perfect Number

At this point, there are many different bounds on odd perfect numbers. These include bounds on the size of the odd perfect number in terms of its number of prime factors, bounds on the size of the largest prime factor, bounds on the size of the smallest component and bounds on the size of N itself. For a given set of positive integers A, we will write $A(x)$ to be the number of elements in A which are at most x. Let E be the set of numbers of Euler's form for an odd perfect number. That is, $n \in E$ if $n = p^a m^2$ where p is prime, $p \equiv a \equiv 1 \pmod 4$, and $(p, m) = 1$. Let P be a given property of a positive integer. We will write E_P to be the set of elements of E satisfying P. We will say that P is a strong property if the density of E_P in E is 0, that is,

$$\lim_{x \to \infty} \frac{E_P(x)}{E(x)} = 0.$$

We will similarly say that P is a weak property if

$$\lim_{x \to \infty} \frac{E_P(x)}{E(x)} = 1.$$

Note for example that for any constant k, all of the following are weak properties:

- "A number must be at least k."
- "A number must have a prime factor at least k."
- "A number must have a component at least k."
- "A number must have at least k distinct primes factors."
- "A number must have at least k total prime factors."

Any finite set of weak properties cannot prove that no odd perfect numbers exist.

However, Ochem and Rao's inequality is in fact a strong property. Define $OR_{\alpha,\beta}(n)$ to be the sentence "$\Omega(n) \geq \alpha\omega(n) + \beta$." It is a not difficult consequence of Theorem 430 in [8] to show the following theorem.

Theorem 9. *Let α and β be real numbers. Assume that $\alpha > 2$. Then $OR_{\alpha,\beta}$ is a strong property.*

We will say that a property P is *substantially stronger* than property Q if two conditions hold:

1. Every element of E which is satisfied by P is satisfied by Q.
2. The set E_P has density zero in the set E_Q. That is,

$$\lim_{x\to\infty} \frac{E_P(x)}{E_Q(x)} = 0.$$

We then strongly suspect that the following is true.

Conjecture 1. Let $\alpha_1, \alpha_2, \beta_1, \beta_2$ be real numbers with $\alpha_1 > \alpha_2 > 2$. Then OR_{α_1,β_1} is substantially stronger than OR_{α_2,β_2}.

Of course, any result of the form "For any odd perfect number N, N must satisfy $OR_{\alpha,\beta}$" cannot by itself resolve the fundamental open question, but we suspect that the strength of Ochem and Rao's result in the sense above is a sign that this is a potentially fruitful direction for further research. We note that something being a strong property does not always line up with our intuition about what should be a "strong" property in a general sense. For example, let $f(x)$ be a function which is increasing for sufficiently large x and satisfying

$$\lim_{x\to\infty} f(x) = \infty.$$

It is not hard to show that the property P_f given by the sentence "For all n, n has a prime factor which is smaller than $f(n)$" is always a weak property. But if one could show that an odd perfect number had to have prime factor always less than $\log\log\log\log n$, that would certainly be noteworthy!

8. Future Work and Related Problems

One major direction for improving these results is to prove there are no triple threats. Proving there are no triple threats would result in substantial tightening of both the bounds for the case when $3 \mid N$ and for the case when $3 \nmid N$. Another natural object of study in this context would be what we call an n-obstruction.

Define an *n-obstruction* to be a set of primes all greater than 3 a_i, b_i, c_i for $1 \leq i \leq n$ and p an odd prime, satisfying for all $1 \leq i \leq n$

1. $\sigma(a_i^2) = p\sigma(b_i^2)\sigma(c_i^2)$
2. $\sigma(b_i^2)$ and $\sigma(c_i^2)$ prime.
3. The a_i are all distinct.

If we can show that a 4-obstruction does not exist, possibly with some very small modulo restrictions we will get a substantially tighter bound. Similarly, if we can rule out 3-obstructions or even a 2-obstruction we would get much tighter bounds (although we suspect that ruling out a 2 is not really doable). Note that at present we cannot even show the following statement which looks like it should be obviously true.

Conjecture 2. There exists some n such there is no odd perfect number N with an n-obstruction a_i, b_i, c_i and satisfying $a_i^2||N$, $b_i^2||N$ and $c_i^2||N$.

Of course, as we improve the linear term in the bounds, the general price paid is that we are subtracting more in the constant term. Thus, in the original Ochem and Rao paper, they had a constant of $-31/7$, and in the subsequent paper we had as worst case constant $-39/8$. One of the original goals of Ochem and Rao's original Inequality (1) was to assist in the proving of Inequality (2), and there is interest in proving inequalities of the form

$$\Omega(N) \geq 2\omega(n) + C \tag{114}$$

where C is reasonably large. At present, the best such inequality is that by Ochem and Rao where $C = 51$.

Inequalities of that form require extensive computation. One needs to check many cases with branching in essentially the standard approach to heavy computations to bound odd perfect numbers. However, Ochem and Rao had as one of their conditions to terminate a branch that Equation (1) forced Equation (2). Obviously, that sort of termination will be more common when one has not just a stronger linear term but a stronger constant term. Using the inequalities from this paper to prove inequalities for the form of Inequality (114) would be easier with less negative constants. For specific small values of ω our inequalities will already give slightly better bounds than used here, but other approaches might improve the constants.

One might hope to use the results of Nielsen which bound the actual size of an odd perfect number N in terms of ω. We present here an approach that is too weak to be useful by itself but might be productive with more work. We will restrict this discussion under the assumption where we have both $5|N$ and $11|N$ where this approach is most likely to work. Assume further that we have $\omega = 10$ which is the smallest possible value of ω not yet ruled out. Note that if $5|\sigma(11^{f_{11}})$ or $11|\sigma(5^{f_5})$, then we can already improve our constant term that way, so we will assume that neither of those occurs. In that case we have

$$(5^{f_5})^2(11^{f_{11}})^2 < (5^{f_5})\sigma(5^{f_5})11^{f_{11}}\sigma(11^{f_{11}}) \leq N. \tag{115}$$

Nielsen [13] has proved that if N is an odd perfect number with k distinct prime factors and largest prime divisor P, then

$$10^{12}P^2N < 2^{(4^k)}. \tag{116}$$

Combining Equation (115) with Nielsen's upper bound (116), as well as the fact that the largest prime factor of an odd perfect number must be at least 10^8 by [6] we get that

$$(5^{f_5})^2(11^{f_{11}})^2 10^{28} < 2^{(4^{10})} \tag{117}$$

which when we take logarithms simplifies to

$$(2\log_2 5)f_5 + (2\log_2 11)f_{11} + 28\log_2 10 < 4^{10} \tag{118}$$

which is a linear inequality restricting f_5 and f_{11} but it is much too weak to give a useful restriction for improving the constant.

There appear to be four possible approaches to improving this inequality. The first approach is that one could improve the size of the largest prime factor of an odd perfect number. This is a project that should be undertaken in general since it has been about a decade since the last substantial improvement on this has occurred; more recent algorithmic improvements and computational power may make this a reasonable step. Unfortunately, it is unlikely that such improvement by itself would substantially improve Inequality (117) since the restriction involves the logarithm of the largest prime divisor. The second approach is to improve the size of the largest prime divisor, restricted to some specific range of ω. It seems very likely that with the additional assumption that $\omega = 10$ or even something like $\omega \leq 15$, that one can substantially improve on the lower bound for the largest prime factor. The third possibility is to use Nielsen's general machinery which he used to prove Equation (116) to incorporate specific prime powers. The fourth possibility is to improve the second inequality in Equation (115) by making precise the intuition that there should be a large part of N which is not included in $\sigma(5^{f_5})\sigma(11^{f_{11}})$. This last looks to be the most promising. However, given how weak Equation (117) is, it will likely require multiple of these approaches for it to be at all productive. Even if one improves it enough to be useful for small values of ω, it will still be likely too weak to be useful for even slightly larger values of ω. Luckily, all four of these approaches would be of general interest to understanding odd perfect numbers. A slightly different approach to Nielsen's bound may also be valid. Again restricting to the situation where $5|N$ and $11|N$, we have

$$5^{f_5}11^{f_{11}}7^{2s}7^{4t}7^{6u}q^e < N \tag{119}$$

and then proceed as before. This inequality is also still too weak to be directly useful by itself but may be combined with bounds on the size of q.

A major part of our improvement in the case when $3 \nmid N$ depended on a specific coincidental factorization of a specific composition of cyclotomic polynomials. Further understanding of such compositions may be relevant for further understanding of odd perfect numbers. These questions about cyclotomic polynomials may be of interest independent of anything involving perfect numbers. We have a conjecture that essentially says that we cannot often get so lucky that we frequently have such factorizations. In particular, we have,

Conjecture 3. Let p and q be distinct odd primes and let $\Phi_p(x)$ and $\Phi_q(x)$ be the pth and qth cyclotomic polynomials. Then at least one of $\Phi_p(\Phi_q(x))$ or $\Phi_q(\Phi_p(x))$ is irreducible.

We also suspect that, in some suitable sense, such compositions being reducible should occur on a set of density zero. In particular, call an ordered pair of positive integers (m, n) to be a *good pair* if $\Phi_m(\Phi_n(x))$ factors over the integers where Φ_m and Φ_n are the mth and nth cyclotomic polynomials. Then we strongly suspect that good pairs are rare in the following sense.

Conjecture 4. Let $D(t)$ count the number of good pairs with both $m \leq t$ and $n \leq t$. Then we have

$$\lim_{t \to \infty} \frac{D(t)}{t^2} = 0.$$

Moreover, we have the following even stricter version: let $f(t)$ and $g(t)$ be strictly increasing functions which go to infinity as t goes to infinity, and let $D_{f,g}(t)$ count the number of good pairs with $m \leq f(t)$ and $n \leq g(t)$. Note that in particular $D(t) = D_{t,t}(t)$. Then we have the following stronger conjecture.

Conjecture 5. For any such $f(t)$ and $g(t)$ we have

$$\lim_{t \to \infty} \frac{D_{f,g}(t)}{f(t)g(t)} = 0.$$

We are uncertain if Conjecture 5 is true, but suspect that if it is true, proving it will be very difficult. We can make corresponding versions of Conjectures 4 and 5 that are restricted to cyclotomic polynomials arising from primes. Define $\bar{D}(t)$ to be the same as $D(t)$ but counting only the good pairs (m, n) where m and n are both prime. Define $\bar{D}_{f,g}(t)$ similarly. We expect the following two conjectures.

Conjecture 6. We have
$$\lim_{t\to\infty} \frac{\bar{D}(t)}{\pi(t)^2} = 0.$$

Conjecture 7. For any such $f(t)$ and $g(t)$ we have
$$\lim_{t\to\infty} \frac{\bar{D}_{f,g}(t)}{\pi(f(t))\pi(g(t))} = 0.$$

Note that similar questions have been asked and answered about general polynomials. See in particular [22] and [18].

Ochem and Rao type results also show that many prime divisors of an odd perfect number must have many repeated prime factors. It is therefore of interest whether this sort of result can be used to improve on results like [12] which rely heavily on inducting on the divisors of an odd perfect number. One other obvious question is whether anyone can replace the Ochem and Rao type results with a better than linear inequality. The methods used in this paper do not seem to have any hope of doing so, but it is plausible that sieve theoretic methods could result in some similar type of restriction. One obvious question is how well we can upper bound $\Omega(N)$ in terms of $\omega(N)$. Recall Nielsen's result [12] that if N is an odd perfect number, then
$$N < 2^{4^{\omega(N)}}. \tag{120}$$
If N is an odd perfect number, then we trivially have $3^{\Omega(N)} < N$, which when combined with Inequality (120) gives
$$\Omega(N) < 4^{\omega(N)} \frac{\ln 2}{\ln 3}.$$
Improving this bound directly in a non-trivial fashion seems worth exploring. Nielsen [12] also showed that if N is an odd perfect number, and we have $P = \prod_{p|N} p$, then
$$N < P^{2^{\omega(N)}}, \tag{121}$$
from which it follows that we have $a_i \leq 2^{\omega(N)-2}$ for at least one of the a_i. It may be possible to use this fact to improve the Ochem-Rao results further. We may also combine the Ochem and Rao type bounds to get a straightforward upper bound for N in terms of ω. In particular, if we know that $\Omega \geq \alpha\omega + \beta$, then we easily have from Inequality (120) that
$$N < 2^{4^{\left(\frac{\Omega-\beta}{\alpha}\right)}}. \tag{122}$$

Using our main theorem we have the result that if $(3, N) = 1$ that

$$N < 2^{4^{\left(\frac{113\Omega+286}{302}\right)}}. \tag{123}$$

It seems worth wondering if we can obtain upper bounds on N in terms of Ω which are substantially better than simply combining the Nielsen bound with the best available Ochem and Rao type bound.

Ochem and Rao also used similar techniques in their proof that an odd perfect number must have a component of size at least 10^{62} [15]. In particular, they first showed that any odd perfect number N must either have a component of size greater than 10^{62} or that N cannot be divisible by any prime less than 10^8. They then concluded that an odd perfect number with all components smaller than 10^{62} can only have primes raised to the first, second, fourth or sixth powers. They obtained a set of linear inequalities relating how many such primes there were which were too tight and thus obtained a contradiction. It is likely that the type tightened bounds in [23] and this paper can be used to improve that type of bound.

An additional area of interest may be to generalize the Ochem and Rao type of results beyond odd perfect numbers. Recall that a number N is said to be multiply perfect if $Nk = \sigma(N)$ for some k, and we then say that N is k-perfect. (Some authors use the term multiperfect rather than multiply perfect.) Under this terminology, perfect numbers are precisely the 2-perfect numbers. It is a long-standing question if the only multiply perfect odd number is 1. We suspect that the Ochem and Rao type results can be extended to odd multiply perfect numbers where the constant term is allowed to be a function of k.

A different generalization of perfect numbers leads to Ore harmonic numbers. Ore noted that if N is a perfect number, then one must have $\sigma(N)|(N\tau(N))$ where $\tau(N)$ is the number of positive divisors of N. Ore called numbers n satisfying $\sigma(n)|(n\tau(n))$ harmonic numbers since they are precisely the numbers where the harmonic mean of their positive divisors is an integer. Note that there are multiply perfect numbers which are not Ore harmonic numbers. Similarly, there are Ore harmonic numbers which are not multiply perfect numbers. Ore asked if all Ore harmonic numbers are even. It would be interesting to see if one can extend the Ochem and Rao type results to Ore harmonic numbers. One can also generalize Ore's harmonic numbers. We will call n a generalized harmonic number if n satisfies $\sigma(n)|(n(\tau(n))g^m)$ where m is some integer and g is the largest odd divisor of $\tau(n)$. It again appears that all solutions here are odd, although as far as we are aware, this generalization has not been investigated in the literature. It would be interesting to see if Ochem and Rao type of bounds can be extended to these generalized harmonic numbers.

Another possible direction is rather than to generalize, instead to narrow the situation. Colton [2] has shown that no perfect number (whether even or odd) satisfies $\tau(n)|n$. However, the set of positive integers which satisfy $\tau(n)|n$ has density zero

[10]. In contrast, the set of numbers n where $\tau(n)|\sigma(n)$ has density 1 [1]. It is not hard to show that the only even perfect number n satisfying $\tau(n)|\sigma(n)$ is $n = 6$. One might ask if we can say anything interesting about odd perfect numbers N satisfying $\tau(N)|\sigma(N)$. In particular, it is likely that Ochem-Rao type results can be substantially improved if one is restricted to this set.

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