Research Statement

Marius Ionescu

I present in the first part of this statement (the first five pages) an overview of my research accomplishments and some projects that I am currently pursuing or plan to pursue in the future. The second part of this document (pages 6 through 28) contains a more detailed and technical description of my past, current, and future research goals, as well as the references that I cite in the statement.

1. An overview of my research

My research interests are in the field of harmonic and functional analysis. In particular, much of my work over the past few years had two main foci: the theory and applications of Calderón-Zygmund and pseudodifferential operators on fractals; and the study of the interaction between operator theory/algebras and fractals, irreversible dynamical systems, the representation theory of groupoids and the theory of wavelets.

Key to most of my research is the definition of an iterated function and its invariant set due to Hutchinson [53]. An iterated function system (i.f.s.) is a collection $(F_1, \ldots, F_N)$ of contractions defined on a complete metric space Y. For most of the applications of our theory we let $Y = \mathbb{R}^n$ for some $n \geq 1$. Given such an i.f.s. there is a unique compact invariant set K such that

$$K = F_1(K) \bigcup \cdots \bigcup F_N(K).$$

The invariant set K is also called a self-similar set. Two important examples of self-similar sets are the unit interval and the Sierpinski gasket. Given a list of probabilities $(\mu_1, \ldots, \mu_N)$ there is a unique invariant measure μ whose support is K such that

$$\mu(A) = \sum_{i=1}^{N} \mu_i \mu(F_i^{-1}(A)), \text{ for all Borel sets } A.$$

I begin first by describing my recent work on harmonic analysis on fractals. The common topic of these papers is the study of differential and pseudodifferential operators on a class of fractals called the post-critically finite (p.c.f.) self-similar sets ([53],[76],[136]) and other measure metric spaces. The underlying technology is the construction of a Laplace operator and Dirichlet form on fractals due to Kigami [76]. Specifically, Kigami used graph approximations for some self-similar sets, including the unit interval and the Sierpinski gasket, to build a self-similar energy form $\mathcal{E}$, that is a Dirichlet form that satisfies

$$\mathcal{E}(u) = \sum_{i=1}^{N} r_i^{-1} \mathcal{E}(u \circ F_i),$$

for some weights $(r_1, \ldots, r_N)$. The Laplacian is defined weakly: $u \in \operatorname{dom} \Delta$ with $\Delta u = f$ if

$$\mathcal{E}(u, v) = -\int_X f v d\mu$$

for all $v \in \operatorname{dom} \mathcal{E}$ with $v|_{V_0} = 0$, where V_0 is the *boundary* of the self-similar set. We assume that f is in $L^2(\mu)$, which gives a Sobolev space.

The results that I proved with my coauthors in [59] represent my first contribution to the literature of analysis on fractals. In this paper, we provide a concrete, self-similar description of the resolvent of the Laplacian on a p.c.f. fractal extending the construction of the Green function. That is, we constructed a symmetric function $G^{(\lambda)}(x,y)$ which weakly solves $(\lambda - \Delta)^{-1}G^{(\lambda)}(x,y) = \delta(x,y)$, meaning that

$$\int_X G^{(\lambda)}(x,y)u(y)d\mu(y) = (\lambda\mathbb{I} - \Delta)^{-1}u(x).$$

We provide also a new view of the resolvent on the unit interval and worked out a few examples, including the Sierpinski gasket. We plan to use our results to study spectral operators of the form

$$\xi(\Delta) = \int_\Gamma \xi(\lambda)(\lambda\mathbb{I} - \Delta)^{-1}d\lambda$$

in a similar manner as used by Seeley [[120],[119]] for the Euclidean situation.

In a joint work with Luke Rogers [60], we proved a characterization of the Calderón-Zygmund operators on fractals. Recall ([126]) that an operator T bounded on $L^2(\mu)$ is a *Calderón-Zygmund operator* if T is given by integration with respect to a kernel $K(x,y)$, that is $Tu(x) = \int_X K(x,y)u(y)d\mu(y)$ for almost all $x \notin \operatorname{supp} u$, such that $K(x,y)$ is a function off the diagonal and such that

$$\left|K(x,y)\right| \lesssim R(x,y)^{-d} \text{ and}$$
$$\left|K(x,y) - K(x,\overline{y})\right| \lesssim \eta\left(\frac{R(y,\overline{y})}{R(x,\overline{y})}\right) R(x,y)^{-d},$$

whenever $R(x,\overline{y}) \geq cR(y,\overline{y})$ for some Dini modulus of continuity η and some $c > 1$. R is a metric on the self-similar set called the resistance metric; it coincides with the Euclidean metric on $[0,1]$. The main result of the paper proves that if T is an bounded operator on $L^2(\mu)$ that has a kernel $K(x,y)$ that is a smooth function off the diagonal of $X \times X$ and satisfies

$$\left|K(x,y)\right| \lesssim R(x,y)^{-d}$$
$$\left|\Delta_y K(x,y)\right| \lesssim R(x,y)^{-2d-1},$$

then the operator T is a Calderón-Zygmund operator. We showed that the *purely imaginary* Riesz and Bessel potentials satisfy these hypothesis. These operators are the first explicit examples of Calderón-Zygmund operators on fractals. Our proofs are quite technical as compared with the Euclidean case due to the lack of a mean value theorem on fractals. We exploit heavily the self-similar structure of the fractals.

In a recent paper joint with Luke Rogers and Robert Strichartz ([61]) we define and study pseudodifferential operators on fractals and other metric measure spaces that include the p.c.f. self-similar sets and Sierpinski carpets. Given a smooth function p such that

$$\left|\left(\lambda\frac{d}{d\lambda}\right)^k p(\lambda)\right| \leq C_k(1+\lambda)^{\frac{m}{d+1}}$$

where d is the Hausdorff dimension with respect to the resistance metric on the fractal, we define a pseudo-differential operator of order m with constant coefficients to be

$$p(-\Delta)u = \int_0^\infty p(\lambda)dP(\lambda)(u).$$

One of the main results of the paper states that if $m = 0$ then $p(-\Delta)$ is given by integration with respect to a kernel that is smooth off the diagonal and satisfies specific decay properties. In particular, the pseudo-differential operators of order 0 are Calderón-Zygmund operators. Our results capture many known results in the classical harmonic analysis. The proofs, however, are different due the interesting fact that, on many fractals, the product of two smooth functions is not smooth anymore [11]. Therefore, standard techniques used in harmonic analysis such as multiplication with smooth bumps are not available to us. Key to our proofs are the sub-gaussian heat kernel estimates that are true on many self-similar sets and other metric measure spaces. One of the main motivation behind our work was the desire to understand elliptic and hypoelliptic operators on fractals. There were many open questions about them in the literature ([136, 13, 114]). We characterize these operators in one of the applications that we describe in the paper. Moreover, we study an interesting class of operators, the so called *quasielliptic* operators, that does not have an analogue in the Euclidean case. For an example of a quasielliptic operator, consider a fractal K with *spectral gaps* such as the Sierpinski gasket. Assume that α, β are positive real numbers such that $\alpha < \beta$ and $\lambda/\lambda' \notin (\alpha, \beta)$ for all λ, λ' in the spectrum of $-\Delta$. Let $a \in (\alpha, \beta)$. Then $q(\lambda_1, \lambda_2) = \lambda_1 - a\lambda_2$ is a quasielliptic symbol. We show that every quasielliptic pseudodifferential operator is equal to an elliptic pseudodifferential operator, though there are quasielliptic differential operators which are not elliptic as differential operators. Other applications that we discuss are the Hörmander type operators and the beginning of the study of wavefront sets on fractals. We began also the study of pseudo-differential operators with variable coefficients, that is operators defined by symbols of the form $p(x, \lambda)$. These operators behave, in general, differently than their counterpart in the Euclidean case, because the product of two *smooth* symbols is not smooth anymore. We managed to prove so far that these operators are bounded on L^p. Moreover, we conjecture that they are Calderón-Zygmund operators. Our results open many new venues of research and provide new tools that will be helpful in the study of specific differential equations of fractals, such as the Scrödinger equation. We plan to pursue this study in the near future.

In a related paper that is joint with Luke Rogers and Alexander Teplyaev ([62]) we study derivations and Fredholm modules on metric spaces with a local regular conservative Dirichlet form. Derivations on p.c.f. fractals and, more generally, C^*-algebras have been studied by Cipriani and Sauvageot in [16] and [17]. We give a concrete description of the elements of the Hilbert module of Cipriani and Sauvageot in the setting of Kigami's resistance forms on finitely ramified fractals [75], a class which includes the p.c.f. self-similar sets studied in [17] and many other interesting examples [1] [139], [112]. We also discuss weakly summable Fredholm modules (an abstract version of an order zero elliptic pseudodifferential operator in the sense of Atiyah [6]) and the Dixmier trace in the cases of some finitely and infinitely ramified fractals (including non-self-similar fractals) if the so-called spectral dimension is less than 2. Even though our results provide information about the "commutative" geometry on fractals, we use heavily noncommutative techniques due to Connes [19] and Cipriani and Sauvageot [16]. A few venues of continuing this research are the extension of our results to noncommutative Dirichlet forms, and the possible generalization of our results to other type of fractals, like the diamonds fractal and Laakso spaces.

I depict in the second part of this overview some of my results that describe the interaction between fractals and operator algebras, as well as results about the fine ideal structure of specific C^*-algebras. In the three papers [55], [54], and [63] that resulted from my thesis I studied the structure of different operator algebras attached to a large class of fractals. These fractals arise from Mauldin-Williams graphs [86] also known as graph directed Markov systems [84]. They are a generalization of iterated function systems described above in the following sense: let $G = (E^0, E^1, r, s)$ be a *finite* directed graph. A *graph directed Markov system* (G.D.M.S.) associated to G consists of a collection $\{T_v\}_{v\in E^0}$ of compact metric spaces, one for each vertex of the graph, and a collection $\{\phi_e\}_{e\in E^1}$ of contractive maps, one for each edge of the graph ([86], [35]). We associate with such a system a C^*-correspondence $\mathcal{X}$ over the C^*-algebra $A = C(T)$, where $T = \bigsqcup T_v$. I built different operator algebras associated with a G.D.M.S. via the Pimsner construction [102] and showed how they capture the dynamics of the fractals. The first main result of [55] states that if the underlying graph has nor sources or sinks then the C^*-algebra associated to the fractal is isomorphic to the Cuntz-Krieger algebra [23] associated to the underlying graph. A theorem that follows from my proof states that a natural generalization of G.D.M.S. to the noncommutative setting is illusory. I showed, however, in [54] that the tensor algebra, a non-self adjoint algebra, that I associated to a G.D.M.S. is locally a complete conjugacy invariant. My results stands in a long series of results that were inspired by Arveson's discovery [5] of the relation between conjugacy invariants for measure preserving transformations and non-self-adjoint operator algebras.

In [63], Watatani and I associate a different C^*-algebra to a G.D.M.S. and we show that it captures more clearly the underlying dynamics. Our approach puts more emphasis on the so called "branch points" of the system, the points that arise due to the failure of the injectivity of the coding by the Markov shift for the graph G. We prove that, in general, these C^*-algebras are quite different from the underlying graph C^*-algebras. Moreover, we show that the K-theory completely characterizes our C^*-algebras and we compute the K-groups for a few examples. In [58] we extend some of these results to C^*-algebra associated to general Markov-Feller operators. In our setting, a Markov-Feller operator is a positive unital map on $C(X)$ for some compact space X. Our construction generalizes a large number of C^*-algebras associated to dynamical systems in the literature. The first main result of [58] describes the simplicity of C^*-algebras associated to Markov operators in terms of the probabilistic properties of these operators. We apply our results to recover known results in the literature as well as provide new applications. In a second theorem, we provide a characterization of the topological quivers ([91]) that arise from our construction. Specifically, we proved that the C^*-algebra of a topological quiver whose vertex and edge spaces are compact and has no infinite emitters is isomorphic to the C^*-algebra of a Markov-Feller operator. We plan to continue our study of these C^*-algebras and better understand the relationship between the support of a Markov operator and the corresponding operator algebras, as well as to decipher the influence of the branch points of Markov operators on the K-theory and the KMS-states of the C^*-algebra. For example, if the state space of a Markov operator is finite, than the support of the operator completely characterizes the C^*-algebra in the sense that the C^*-algebras associated with two Markov operators are the same if and only if their support is the same. This statement is, most likely, not true for a general Markov-Feller operator. However, we conjecture that two operators with the same support give rise to Morita equivalent C^*-algebras.

A closely related project that I am currently undertaking with Paul Muhly relates the theory of wavelets to natural representations of groupoids attached to local homeomorphisms. The main tool that we use is what we call the Renault-Deaconu groupoid ([105],[108],[28]) associated to a local homeomorphishm:

$$G = \{(x, n, y) \in X \times \mathbb{Z} \times X \ : \ T^k(x) = T^l(y), n = k - l\},$$

endowed with a suitable topology such that G becomes an étale, locally compact groupoid. Some partial results and examples have appeared in [57]. The main example of the paper describes how one can recover the classical wavelet analysis [83] via the "trivial" representation of the Renault-Deaconu groupoid associated to the map $T(z) = z^2$ on the unit circle. We are currently extending this work and we proved recently a general theorem that provides a unitary extension to endomorphisms associated with local homeomorphisms, generalizing a number of results in the literature (see, for example, [80]).

Extending our analysis of Renault-Deaconu groupoids, Alex Kumjian and I study in [56] the connection between the Hausdorff measure on a compact space X and the KMS states on the C^*-algebras attached to local homeomorphisms that satisfy a "local scaling condition". Roughly speaking, a local homeomorphism satisfies the local scaling condition if, locally, the map is a similitude. We prove that, for such a local homeomorphism, the Hausdorff measure on the underlying space gives rise to a KMS state on the C^*-algebra of the groupoid. Moreover, we provide conditions that guarantee that the KMS-states are unique and have a unique inverse temperature. We describe how our results provide a quick computation of the topological entropy for many local homeomorphisms. We extend our results to an example of a local homeomorphism on a fractafold [132] that does not satisfy the hypothesis of our main theorem. We believe that this is the first example of a local homeomorphism on spaces built out of fractals and it was one of the main motivation behind our work. We view the local homeomorphism that we defined as the fractal analogue of the map $T(z) = z^2$ on the torrus.

In ongoing work, Kumjian and I use actions of Renault-Deaconu groupoids on spaces to study symmetries of the so called "fractafolds". These are spaces that were introduced by Strichartz [131] to mimic the relationship between the unit interval and the real line in the fractal world. Given an invariant set K of an iterated function system with N maps, each infinite sequence x in $\{1, \dots, N\}^\infty$ gives rise to a fractal blowup by taking the preimage inverse of K in the order given by x. Strichartz noticed that two such fractal blowup are homeomorphic if and only if the parametrizing sequences are, eventually, the same. Based on this observation, we assemble these fractal blowups into a bundle on which the Cuntz groupoid, a particular example of a Renault-Deaconu groupoid, acts. We proved so far that this action gives rise to a natural local homeomorphism on the bundle that is essentially free. Moreover, we are studying various properties of the action groupoid and its C^*-algebra.

In a series of papers done jointly with Dana P. Williams, we undertake the task of studying the fine structure of the ideal space of groupoid and Fell bundle C^*-algebras [65, 64, 66, 67, 68]. In [64] we prove the so called Effros-Hahn conjecture ([36]) for groupoid C^*-algebras: every primitive ideal in the C^*-algebra of an amenable groupoid is induced from a stability group. Our results provide a significant sharpening of some results in the literature [107]. Our proof is quite technical and even though we proved our results for groupoid C^*-algebras, we use heavily the theory of groupoid dynamical systems as in [106] and [107]. We show in [65] that the induced representation of an irreducible representation of a stability group is irreducible. This result generalizes a number of theorems in the literature and is one of the main pillars in our proof of the Effros-Hahn conjecture.

In [67] begin the systematic study of the primitive ideal space of C^*-algebra associated to Fell bundles over groupoids [95]. Fell bundles generalize most known examples of dynamical systems: groupoids [105] and twisted groupoids [78], C^*-dynamical systems [100] and Green's twisted dynamical systems [43], and (twisted) groupoid dynamical system [106]. We prove in [67] that if $p : \mathcal{A} \to G$ is a Fell bundle over a groupoid then there is a natural continuous action of the groupoid on the primitive ideal space of the C^*-algebra A sitting over the unit space of a Fell bundle. We use this action to generalize a result about short exact sequences to Fell bundles over groupoids. Namely, we prove that if I is a G-invariant ideal in A then one can build a "reduction" of the Fell bundle to I and a "quotient" Fell bundle by I and we obtain a short exact sequence. Our main result in [66] extends a classic Morita Equivalence Result of Green's [43] to the C^*-algebras of Fell bundles over groupoids. The main results states that if H is a stability group of G, then $C^*(G, \mathcal{A})$ and $C^*(H, \mathcal{A}|_H)$ are Morita equivalent. Green's result is a particular case of an application of our result to the case when G is a group. Our main result in [68] states that, for a Fell bundle C^*-algebra, the induced representation of a an irreducible *homogeneous* representation is irreducible. This results is a generalization of results in [33],[117],[118] and it constitutes an important step towards the proof of a generalized Effros-Hahn conjecture for Fell bundle C^*-algebras. Our proof requires an intermediate result which is of considerable interest on its own. Namely, the induced representation of an irreducible representation over a *stability group* is irreducible. We plan to continue our project in joint work with Dana Williams. Success in proving the Effros-Hahn conjecture in this context will provide important information about the primitive ideal space and simplicity of Fell bundle C^*-algebras and, in particular, will provide information about the ideal structure of many C^*-algebras associated to dynamical systems.

Extended Research Statement

Marius Ionescu

As I mentioned at the beginning of my research statement, I provide in what follows a more detailed and technical description of my research accomplishments, of some problems that I am currently pursuing, and of directions for future research.

2. Harmonic analysis on fractals

I became interested in problems related to analysis on fractals during the year I spent at Cornell University in 2007-2008 as a visiting assistant professor. There I had the opportunity to interact with many people working in the area of harmonic analysis and differential operators on fractals, such as Strichartz, Kigami, Teplyaev, and Rogers. I have also been involved in the research of the students participating in the R.E.U. program in fractal analysis at Cornell. This fact helped me understand the many venues of research in this mathematical area that are available to me, as well as to undergraduate and graduate students. I became immediately involved in some interesting projects on the subject. I will describe next a few projects that I have recently finished, I am currently working on or intend to begin working on in the near future.

An iterated function system (i.f.s.) is a collection $\{F_1, \ldots, F_N\}$ of contractions on a complete metric space X. For such an i.f.s. there exists a unique invariant (self-similar) compact set K ([53],[10],[34]). That is,

$$K = F_1(K) \bigcup \cdots \bigcup F_N(K).$$

If $\omega_1, \omega_2, \ldots \omega_n \in \{1, \ldots, N\}$, then we say that $\omega = \omega_1\omega_1 \ldots \omega_n$ is a word of length n over the alphabet $\{1, \ldots, N\}$. The subset $K_\omega = F_\omega(K) := F_{\omega_1} \circ \cdots \circ F_{\omega_n}(K)$ is called a cell of level n. We denote by W^n the set of all words of length n and by W^* the set of *all* finite words over $\{1, \ldots, N\}$. We write W^∞ for the set of infinite words (sequences) with elements in $\{1, \ldots, N\}$.

Since the map F_i, $i = 1, \ldots, N$, is a contraction, it follows that it has a unique fixed point x_i for all $i = 1, \ldots, N$. We say that K is a *post-critically finite (p.c.f.) self-similar* set if there is a subset $V_0 \subseteq \{x_1, \ldots, x_N\}$ satisfying

$$F_\omega(K) \bigcap F_{\omega'}(K) \subseteq F_\omega(V_0) \bigcap F_{\omega'}(V_0)$$

for any $\omega, \omega' \in W^*$ that have the same length and $\omega \neq \omega'$. The set V_0 is called the *boundary* of K and the boundary of a cell K_ω is $F_\omega(V_0)$. One defines $V_1 = \bigcup_i F_i(V_0)$, and, inductively, $V_n = \bigcup_i F_i(V_{n-1})$ for $n \geq 2$. Then the fractal K is the closure of $V^* = \bigcup_n V_n$.

Two important examples of p.c.f. self-similar sets are the unit interval $[0, 1]$ and the Sierpinski gasket ([76, 136, 40, 11, 97, 132, 140]). The maps $F_1(x) = 1/2x$ and $F_2(x) = 1/2x + 1/2$ form an i.f.s. on $\mathbb{R}$ with invariant set $[0, 1]$. Then $V_0 = \{0, 1\}$ and V^* consists of the dyadic points. To study the Sierpinski gasket one can consider three points A_1, A_2, and A_3 in $\mathbb{R}^2$ and define $F_i(x) = 1/2(x - A_i) + A_i$, $i = 1, 2, 3$. Then (F_1, F_2, F_3) is an i.f.s. on $\mathbb{R}^2$ and its invariant set is the Sierpinski gasket. In this case $V_0 = \{A_1, A_2, A_3\}$. Other examples of p.c.f. fractals that fall under the scope of our results are the affine nested fractals [39] that in turn are generalizations of the nested fractals [82] and include the pentagasket and Lindstrøm's snowflake, the Vicsek sets [87], and the abc-gaskets [71].

A theory of analysis on certain p.c.f. self-similar fractals has been developed around the Laplace operator by Kigami ([76]; see also [136]). The Laplacian on many p.c.f. fractals may be built using Kigami's construction from a self-similar Dirichlet energy form $\mathcal{E}$ on K with weights $\{r_1, \ldots, r_N\}$:

$$\mathcal{E}(u) = \sum_{i=1}^{N} r_i^{-1} \mathcal{E}(u \circ F_i).$$

The existence of such forms is non-trivial, but on a large collection of examples they may be obtained from the approximating graphs as the appropriate renormalized limit of graph energies ([76, 136]). The second ingredient is the existence of a unique self-similar measure $\mu(A) = \sum_{i=1}^{N} \mu_i \mu(F^{-1}(A))$, where $\{\mu_1, \ldots, \mu_N\}$ are weights such that $0 < \mu_i < 1$ and $\sum \mu_i = 1$, see [53]. Then the Laplacian is defined weakly: $u \in \operatorname{dom} \Delta$ with $\Delta u = f$ if

$$\mathcal{E}(u, v) = -\int_X f v d\mu$$

for all $v \in \operatorname{dom} \mathcal{E}$ with $v|_{V_0} = 0$. The domain of the Laplacian depends on the assumptions that one makes about f. Kigami assumes that f is continuous, but for our work it will be more natural to assume that f is in $L^2(\mu)$, which gives a Sobolev space.

The *effective resistance metric* $R(x, y)$ on K is defined via

$$R(x, y)^{-1} = \min\{\mathcal{E}(u) \,:\, u(x) = 0 \text{ and } u(x) = 1\}.$$

It is known that the resistance metric is topologically equivalent, but not metrically equivalent to the Euclidean metric ([76, 136]).

Real analysis is performed, however, mainly on the real line and torus and not on the unit interval. Therefore, some of the spaces that we consider are built from p.c.f. fractals as in [131, 132]. In those papers the author defines fractal blowups of a fractal K and fractafolds based on K. The former generalizes the relationship between the unit interval and the real line to arbitrary p.c.f. self-similar sets, while the latter is the natural analogue of a manifold. Let $w \in \{1, \ldots, N\}^\infty$ be an infinite word. The *fractal blowup* associated to w is

$$X = \bigcup_{m=1}^{\infty} F_{w_1}^{-1} \ldots F_{w_m}^{-1} K.$$

Two such blowups are naturally homeomorphic if the parametrizing words are eventually the same. In general there are an uncountably infinite number of blowups which are not homeomorphic. We assume that the infinite blowup X has no boundary. This happens unless all but a finite number of letters in w are the same. One can extend the definition of the energy $\mathcal{E}$, the measure μ, and the Laplacian Δ to X. It is known that, for a large class of p.c.f. self-similar sets including the Sierpinski gasket and nested fractals, the Laplacian on an infinite blowup without boundary has pure point spectrum ([140], [116]).

A *fractafold* [132] based on K is a set for which every point has a neighborhood which is homeomorphic to a neighborhood of a point in K. We will consider fractafolds that consist of a finite or infinite union of copies of K glued together at some of the boundary points. The fractafold X is compact if and only if we consider a finite number of copies of K. We suppose in the following that all the copies of K have the same size in X. If all the boundary points of the copies of K are paired, then the fractafold X has no boundary. One example of a fractafold is the unit circle viewed as two copies of the unit interval glued together at the endpoints; another one is the double cover of the Sierpinski gasket, where one consider two copies of the gasket with corresponding boundary points paired. An explicit description of the spectral resolution of the fractafold Laplacian for certain infinite fractafolds is given in [137].

To develop a theory that resembles the theory of P.D.E., Strichartz extended the definition of the energy and the Laplacian to products of p.c.f. self-similar sets [134], [129]. An important point to keep in mind is that products of p.c.f. self-similar sets are not p.c.f. self-similar sets anymore. Strichartz described in [134] how one can extend the definition of the Laplacian and energy to products of fractals.

From now on, X will stand for either the fractal K or a fractafold built out of K as explained above. It is known ([76],[136]) that X satisfies the doubling condition, that is, there is a constant $C > 0$ such that

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)) \text{ for all } x \in X \text{ and } r > 0.$$

We assume that the heat operator $e^{t\Delta}$ has a positive kernel $h_t(x, y)$ that satisfies the following sub-Gaussian upper estimate

$$h_t(x, y) \leq c_1 t^{-\beta} \exp\left(-c_2 \left(\frac{R(x, y)^{d+1}}{t}\right)^{\gamma}\right), \tag{1}$$

where $c_1, c_2 > 0$ are constants independent of t, x and y. In this expression both d and γ are constants that depend on X and $\beta = d/(d + 1)$. Moreover we assume that $h_z(x, y)$ is a holomorphic function on $\{\operatorname{Re} z > 0\}$. The estimates (1) are known to be true for a large number of p.c.f. self-similar sets ([39], [9], [47], [49], [129]) with R being the resistance metric, the constant d being the Hausdorff dimension with respect to the resistance metric, and γ a constant specific to the fractal. The heat kernel estimates (1) are valid for the Sierpinski carpets[1] ([7, 8]) and Laakso spaces ([128]) as well.

While the main motivation of our work comes from the analysis on fractals, our results capture the known facts from classical harmonic analysis on $\mathbb{R}^n$ and riemannian manifolds and can be used to define pseudodifferential operators on other metric measure space. For a non-fractal example, the operators defined in [50] fall under the hypothesis of our work.

2.1. **Resolvent and Heat Kernels on Fractals.** The heat equation, the heat kernel, and the heat kernel estimates have been central topics in analysis on fractals. These topics have been studied primarily with probabalistic methods [9], [76], [39], [48]. In collaboration with Pearse, Ruan, Rogers and Strichartz, we were able to give an analytic formula for the resolvent of the Laplacian on a p.c.f. fractal that by-passed probabilistic methods [59]. That is, we constructed a symmetric function $G^{(\lambda)}(x, y)$ which weakly solves $(\lambda - \Delta)^{-1} G^{(\lambda)}(x, y) = \delta(x, y)$, meaning that

$$\int_X G^{(\lambda)}(x, y)u(y)d\mu(y) = (\lambda - \Delta)^{-1}u(x).$$

For $\lambda = 0$ our construction recovers the Green function for the Laplacian Δ. Our main theorem provides an explicit description of the resolvent kernel. Heuristically, the resolvent kernel is the sum of the weak solutions of the resolvent problem in all cells of all order. We worked out in detail our construction for a series of examples, including the unit interval and the Sierpinski gasket. Rogers extended our construction and estimates of the resolvent to infinite blow-ups of P.C.F. fractals in [113]. Moreover, using our results, he proved using analytic methods the heat kernel estimates.

We hope that the resolvent kernel will provide other information about spectral operators of the form

$$\xi(\Delta) = \int_\Gamma \xi(\lambda)(\lambda\mathbb{I} - \Delta)^{-1} d\lambda$$

in the same manner as used by Seeley [119][120] for the Euclidean situation. Such study would be significant in providing us information about operators that don't fall under the hypothesis of Calderón-Zygmund theory on fractals, such as the Schrödinger operator on infinite blowups.

[1] Recall that while the Sierpinski carpets are self-similar sets, they are *not* p.c.f self-similar spaces

2.2. **Calderón-Zygmund operators on p.c.f. fractals.** In [60] we give the first natural examples of Calderón-Zygmund operators in the theory of analysis on post-critically finite self-similar fractals. This is achieved by showing that the *purely imaginary* Riesz and Bessel potentials on nested fractals with 3 or more boundary points are of this type. Complex powers of the Laplacian on Euclidean spaces and manifolds and their connection to pseudodifferential operators have been studied intensely (see, for example, [119, 120, 125, 138, 26] and the citations within). Our main focus is to show that the Riesz potentials $(-\Delta)^{i\alpha}$ and the Bessel potentials $(I-\Delta)^{i\alpha}$, $\alpha \in \mathbb{R}$, are Calderón-Zygmund operators in the sense of [126]: an operator T bounded on $L^2(\mu)$ is called *a Calderón-Zygmund operator* if T is given by integration with respect to a kernel $K(x,y)$, that is

$$Tu(x) = \int_X K(x,y)u(y)d\mu(y)$$

for almost all $x \notin \operatorname{supp} u$, such that $K(x,y)$ is a function off the diagonal and such that

$$\big|K(x,y)\big| \lesssim R(x,y)^{-d} \text{ and} \tag{2}$$

$$\big|K(x,y) - K(x,\overline{y})\big| \lesssim \eta\left(\frac{R(y,\overline{y})}{R(x,\overline{y})}\right) R(x,y)^{-d}, \tag{3}$$

whenever $R(x,\overline{y}) \geq cR(y,\overline{y})$ for some Dini modulus of continuity η and some $c > 1$. We say, in this case, that $K(x,y)$ is *a standard kernel*. The operator T is a *singular integral operator* if the kernel $K(x,y)$ is singular at $x = y$. The main result of [60] is:

Theorem 2.1. *Let K be a nested fractal and X be either K or an infinite blow-up of K without boundary. Suppose T is an bounded operator on $L^2(\mu)$ that has a kernel $K(x,y)$ that is a smooth function off the diagonal of $X \times X$ and satisfies*

$$\big|K(x,y)\big| \lesssim R(x,y)^{-d} \tag{4}$$

$$\big|\Delta_y K(x,y)\big| \lesssim R(x,y)^{-2d-1}. \tag{5}$$

Then the operator T is a Calderón-Zygmund operator.

The proof of this theorem for a general p.c.f. self-similar set is considerably more technical compared to the Euclidean case because of the lack of a mean value theorem in the fractal world. We use in a crucial way the self-similarity property of the fractals and the formula for the Green function on fractals ([76]).

We extend our analysis to products of nested fractals and their infinite blowups. We also generalize our results to the so called Laplace type transforms. Recall that a function $p : [0,\infty) \to \mathbb{R}$ is said to be of *Laplace transform type* if $p(\lambda) = \lambda \int_0^\infty m(t)e^{-t\lambda}dt$, where m is uniformly bounded. Then we can define an operator

$$p(-\Delta)u = (-\Delta)\int_0^\infty m(t)e^{t\Delta}udt$$

with a kernel

$$K(x,y) = \int_0^\infty (-\Delta_1)h_t(x,y)m(t)dt. \tag{6}$$

that is smooth off the diagonal and it satisfies the estimates (4) and (5).

Riesz and Bessel potentials for negative real powers in the context of metric measure spaces, including fractals, have been studied in [52] (see also [51]), however their results are not directly applicable in our setting.

We conjecture that the Riesz and Bessel potentials on fractals, as well as the Laplace type transforms, are in fact singular integral operators. We plan to tackle this problem as part of a larger project that I describe next.

2.3. **Pseudodifferential operators on p.c.f. fractals.** Several recent papers have studied properties of spectral operators on fractals. For example, [135] shows some new convergence properties of Fourier series on fractals with spectral gaps and establishes a Littlewood-Paley inequality for such fractals. Numerical results suggest that in addition to the Riesz and Bessel potentials that we studied in [60], other spectral operators on fractals are given by integration with kernels that satisfy estimates as in Theorem 2.1 [3, 21]. In [61] we define and study pseudodifferential operators on metric measure spaces endowed with a non-positive self-adjoint Laplacian such that the heat kernel of the heat operator satisfies sub-Gaussian estimates. Some of the results proved in [61] are extensions of the corresponding results from classical harmonic analysis (see, for example [138, 124, 126]) However the proofs of our results are very different. The main reason for this difference is that the product of smooth functions is, in general, no longer in the domain of the Laplacian [11]. Therefore techniques that are essential in real analysis like multiplication with a smooth bump are not available to us. We frequently use the Borel type theorem proved in [115] to decompose a smooth function in a sum of smooth functions supported in specified cells. We review next the main definitions and results of this paper.

For fixed $m \in \mathbb{R}$ we define the symbol class S^m to be the set of $p \in C^\infty((0,\infty))$ with the property that for any integer $k \geq 0$ there is $C_k > 0$ such that

$$\left|\left(\lambda\frac{d}{d\lambda}\right)^k p(\lambda)\right| \leq C_k(1+\lambda)^{\frac{m}{d+1}}$$

for all $\lambda > 0$, where d is the Hausdorff dimension with respect to the resistance metric on the fractal. The rationale for dividing m by $d+1$ is that the Laplacian behaves like an operator of order $d+1$.

If p is any bounded Borel function on $(0,\infty)$ then one can define an operator $p(-\Delta)$ via

$$p(-\Delta)u = \int_0^\infty p(\lambda)dP(\lambda)(u).$$

This operator extends to a bounded operator on $L^2(\mu)$ by the spectral theorem. If $p \in S^m$ with $m > 0$, then $q(\lambda) := (1+\lambda)^{-\frac{m}{d+1}}p(\lambda)$ is bounded and one can define $p(-\Delta) = (I-\Delta)^{\frac{m}{d+1}}q(-\Delta)$. For fixed $m \in \mathbb{R}$ we define the class ΨDO_m of *pseudodifferential operators with constant coefficients* on X to be the collection of operators $p(-\Delta)$ with $p \in S^m$.

We prove that the pseudodifferential operators with constant coefficients satisfy the symbolic calculus and that they are given by integration with respect to kernels that are smooth and decay off the diagonal, extending some of the results of [60]. The first of the main theorems in the paper is

Theorem 2.2. *Let $p : (0,\infty) \to \mathbb{C}$ be an S^0-symbol. Then $p(-\Delta)$ has a kernel $K(x,y)$ that is smooth off the diagonal of $X \times X$ and satisfies*

$$|K(x,y)| \lesssim R(x,y)^{-d} \tag{7}$$

and

$$|\Delta_x^l \Delta_y^k K(x,y)| \lesssim R(x,y)^{-d-(l+k)(d+1)}. \tag{8}$$

Thus the class of pseudodifferential operators of order 0 on p.c.f. fractals are Calderón-Zygmund operators and they extend to bounded operators on L^q, for all $1 < q < \infty$. In this context we therefore recover the results of [141], [121]. We extend our analysis also to products of metric measure spaces such that the heat kernel on each factor satisfies our main estimates (1). However, we allow the constants in the estimates to be different. Thus, the class of examples for our results is quite large. For instance, one can apply our results to a product between the real line and a fractafold based on the Sierpinski gasket or a Sierpinski carpet.

We believe that actually more is true. Namely, we conjecture that the pseudodifferential operators of order 0 are singular integral operators. This result is known to be true in the real case ([126]). However, the methods of proof used in the real analysis case are not applicable to our context. This is one of the projects that we plan to pursue in the near future.

We define Sobolev spaces on these fractals and prove that pseudodifferential operators with constant coefficients are bounded on them. One of the main applications that we want to work on in the near future is a "Fubini-type" theorem as in [130]:

Conjecture 2.3. *Assume that X^k is a product of k copies of an infinite blowup X of a p.c.f. self-similar set. Let f be a function on X and let f_j be the restriction of f to $k-1$ copies of X obtained by freezing the jth component, $j = 1, \ldots, k$. Then $f \in L^p_s(X^k)$ if and only if $f_j \in L^p(X^{k-1})$ for $j = 1, \ldots, k$, and then*

$$\|f\|_{L^p_s(X^k)} \simeq \sum_{j=1}^{k} \|f_j\|_{L^p(X^{k-1})}.$$

In order to prove this conjecture we need first to enhance our knowledge of *fractional* Sobolev spaces on fractals. Namely, when is a bounded operator on L^p_s or L^p given by a multiplier? What is the correct notion of "commuting with translations" on fractals? A positive answer to our conjecture will be useful in the study of differential equations and differential operators such as the Schrödinger operator that I discuss in a future section. Such operators don't fall under the scope of our Theorem 2.2 and estimating their L^p and L^p_s norms is a difficult task.

Another important application that we discuss in [61] the study elliptic and hypoelliptic operators. Namely, we prove that a pseudodifferential operator satisfies the pseudo-local properties and that an elliptic operator is hypoelliptic. This gives positive answers to some open questions posed in [136, 13, 114].

An interesting class of operators that can be defined on fractals with spectral gaps are the so called *quasielliptic* operators ([13, 121]). For example, assume that α, β are positive real numbers such that $\alpha < \beta$ and $\lambda/\lambda' \notin (\alpha, \beta)$ for all λ, λ' in the spectrum of $-\Delta$. Let $a \in (\alpha, \beta)$. Then $q(\lambda_1, \lambda_2) = \lambda_1 - a\lambda_2$ is a quasielliptic symbol. We show that every quasielliptic pseudodifferential operator is equal to an elliptic pseudodifferential operator, though there are quasielliptic differential operators which are not elliptic as differential operators.

We extend the class of pseudodifferential operators to include operators for which the derivatives of the symbols have a slower rate of decay. Namely, for $0 \le \rho \le 1$ we consider the collection S^m_ρ of symbols $p \in C^\infty((0, \infty))$ with the property that for any $k \ge 0$ there is $C_k(\rho) > 0$ such that

$$\left|\left(\lambda^\rho \frac{d}{d\lambda}\right)^k p(\lambda)\right| \le C_k(\rho)(1+\lambda)^{\frac{m}{d+1}}$$

for all $\lambda > 0$, where $0 \le \rho \le 1$. We proved that if $1/(\gamma+1) < \rho \le 1$ then the kernels of the corresponding pseudodifferential operators are smooth; they are not, however, Calderón-Zygmund operators if $\rho < 1$, and they might not be bounded on $L^q(\mu)$. As an application to our results we consider the Hörmander type hypoelliptic operators. We say that a smooth map $p : (0, \infty)^N \to \mathbb{C}$, where $N \ge 2$, is a *Hörmander type hypoelliptic symbol* if there are $\varepsilon > 0$ and $A > 0$ such that

$$\left|\frac{\frac{\partial^\alpha}{\partial \lambda^\alpha} p(\lambda)}{p(\lambda)}\right| \le c_\alpha |\lambda|^{-\varepsilon|\alpha|} \text{ for } |\lambda| \ge A,$$

where c_α are postive constants for all $\alpha \in \mathbb{N}^N$. We prove that the Hörmander type hypoelliptic operators are hypoelliptic. This extends one side of the classical result ([138, Chapter III, Theorem 2.1]). The converse is false in general, as is exemplified by the quasielliptic operators.

2.4. **Wavefront sets and microlocal analysis.** In another application we introduce in [61] the wavefront set and microlocal analysis on products of compact spaces built out of fractals. Let Γ denote an open cone in $\mathbb{R}_+^N$ and Ω an open set in X. We use φ_{α_k} to denote L^2 normalized eigenfunctions corresponding to eigenvalues λ_{α_k}, and set $\lambda_\alpha = \lambda_{\alpha_1} + \cdots + \lambda_{\alpha_N}$. A distribution u is defined to be C^∞ in $\Omega \times \Gamma$ if it can be written on Ω as a linear combination of eigenfunctions with coefficients having faster than polynomial decay over the eigenvalues in Γ. More precisely, if there is a sequence b_n and a function v with $v|_\Omega = u$ that has the form

$$v = \sum_\alpha c_\alpha \varphi_{\alpha_1} \otimes \varphi_{\alpha_2} \otimes \cdots \otimes \varphi_{\alpha_N}, \tag{9}$$

for values c_α such that $|c_\alpha| \le b_n(1+\lambda_\alpha)^{-n/(d+1)}$ for all n and all $\{\lambda_{\alpha_1}, \dots, \lambda_{\alpha_N}\} \in \Gamma$. We define the *wavefront set* of u, $\mathrm{WF}(u)$, to be the complement of the union of all sets where u is C^∞. If u is a smooth function on X then $\mathrm{WF}(u)$ is empty. More generally, $\operatorname{singsupp} u$ is the projection of $\mathrm{WF}(u)$ onto X. The main theorem that we proved so far about wavefront sets is

Theorem 2.4. *If $p \in S^m$ then $\mathrm{WF}\, p(-\Delta)u \subseteq \mathrm{WF}(u)$. If in addition $p(-\Delta)$ is elliptic then $\mathrm{WF}(p(-\Delta)u) = \mathrm{WF}(u)$.*

Thus pseudodifferential operators may decrease the wavefront set and that elliptic operators preserve the wavefront set, extending results from classical harmonic analysis (see, for example, [122]). We also describe the wavefront set for a few concrete examples. While we can not hope to prove that singularities propagate along the wavefront set, as in the classical real analysis, since for many p.c.f. fractals there are compactly supported eigenfunctions of the Laplacian, we do believe that the wavefront set will play an important role in the analysis of P.D.E. on fractals. They should provide important information about solutions for the wave equations on fractals and other equations. In particular, we believe the following conjecture is true:

Conjecture 2.5. *Let X_1 and X_2 be two cells in X such that $X_1 \subset X_2$ and let U_i be the interior of X_i. Let p be a pseudodifferential operator with constant coefficients. If u satisfies the equation $p(-\Delta)u = 0$ in U_2 and vanishes on U_1 then u must vanish on U_2.*

We plan to pursue the study of wavefront sets on fractals in a project joint with Robert Strichartz.

2.5. **Pseudodifferential operators with variable coefficients and applications.** In the last section of [61], we define and study some properties of pseudodifferential operators with variable coefficients on fractals. The definition of a symbol with variable coefficients is straight forward. Namely, for $m \in \mathbb{R}$ we define the symbol class S^m to consist of the smooth functions $p : X \times (0,\infty) \to \mathbb{C}$ such that for each $k \in \mathbb{N}$ and $j \in \mathbb{N}$ there is a positive constant $C_{j,k}$ such that

$$\left| \left(\lambda \frac{\partial}{\partial \lambda}\right)^k \Delta_x^j p(x,\lambda) \right| \le C_{j,k}(1+\lambda)^{\frac{m}{d+1}}. \tag{10}$$

Then we define the operator class of pseudodifferential operators with variable coefficients ΨDO_m by

$$p(x,-\Delta)u(x) = \int\int p(x,\lambda) P_\lambda(x,y) u(y)\, dy\, d\lambda,$$

for $p \in S^m$ and $u \in \mathcal{D}$. Recall that for a large class of fractals the domain of the Laplacian is not closed under multiplication [11]. This implies in our case that the symbolic calculus is not valid for the pseudodifferential operators with variable coefficients. Namely, the product of two symbols in the above sense is no longer smooth and it can not be a symbol of a pseudodifferential operator. Another consequence of this fact is that the kernel of these operators cannot be smooth. Nevertheless we managed to prove the following important result in the case when X is *compact* and has no boundary.

Theorem 2.6. *Suppose that X is a compact fractafold with no boundary and $p \in S^0$. Then the operator $Tu(x) = p(x, -\Delta)u(x)$ is given by integration with a kernel that is continuous off the diagonal and satisfies the estimate*

$$|K(x,y)| \lesssim R(x,y)^{-d}. \tag{11}$$

While the estimates of the previous theorem alone do not allow us to apply our [60, Theorem 3.2] to conclude that $p(x, -\Delta)$ are Calderón-Zygmund operators, we managed to prove the following theorem:

Theorem 2.7. *Suppose that X is a compact fractafold with no boundary and $p \in S^0$. Then the operator $Tu(x) = p(x, -\Delta)u(x)$ extends to a bounded operator on $L^q(\mu)$ for all $1 < q < \infty$.*

These results cannot be obtained using the results of [141] and [121]. Our method of proof consists in approximating $p(x, -\Delta)$ by an infinite sum of *constant* coefficient pseudodifferential operators that are *uniformly* bounded on $L^q(\mu)$. Even though we managed to prove Theorem 2.7 without the use of the Calderón-Zygmund theory, the following conjecture is important in understanding some differential equations:

Conjecture 2.8. *Suppose that X is a compact fractafold without boundary and $p \in S^0$. Then the operator $Tu(x) = p(x, -\Delta)u(x)$ is a Calderón-Zygmund operator.*

We already know that T is bounded on $L^2(\mu)$ and that its kernel satisfies the estimate (2). In order to prove that the kernel K of the operator T satisfies (3) as well we plan to use some of our estimates from [62]. Once the above conjecture is proved, then the next step is to prove the following conjecture:

Conjecture 2.9. *Suppose that X is a compact fractafold without boundary and $p \in S^0$. Then the operator $Tu(x) = p(x, -\Delta)u(x)$ is a singular integral operator on $L^2(\mu)$.*

We want to pursue this idea even further. Namely, we want to investigate whether the Theorems 2.6 and 2.7 as well as the Conjectures 2.8 and 2.9 are valid for *non-compact* fractafolds. As a first step in proving these conjectures in the non-compact case, we need to prove that the operator $p(x, -\Delta)$ is bounded on $L^2(\mu)$. The extension of this result from the compact case to infinite blowups is far from trivial, since we can not multiply by smooth cut-off functions. We plan to use the results in [115] and decompose a smooth functions into a sum of compactly supported smooth functions. We already know that the pseudodifferential operator associated to each such function is bounded on $L^2(\mu)$. The main technical point is to prove that the bounds are summable.

A positive answers to the conjectures described above would allow us to use pseudodifferential techniques in the study of various differential equations on fractals. In particular, we plan to study Schrödinger operators of the form $H = -\Delta + \chi$, where χ is a potential. The authors of [98] showed that, on the Sierpinski gasket, the spectrum of a Schrödinger operator associated to a continuous potential χ breaks into clusters whose asymptotic distribution may be described precisely. The authors use the spectral decimation method for the Sierpinski gasket to prove their main results. However, similar results are true for the Schrödinger operators corresponding to the Laplace-Beltrami operator on a compact riemannian manifold, with smooth potential ([46],[45][44],[142]). The proof of the results for riemannian manifolds uses extensively pseudodifferential techniques. Therefore, not only we plan to use our theory of pseudodifferential operators and show that the results of [98] are true for all the p.c.f. self-similar sets for which the heat kernel satisfies the estimate (1), but also we plan to provide a more detailed description of the spectrum for Schrödinger operators with *smooth* potentials. An important problem that we need to overcome is the fact that the class of pseudodifferential operators with variable coefficients does not satisfy the symbolic calculus. Therefore the methods of [142] need to be changed significantly in our context. The main tool that we want to use again is the approximation of a variable coefficient pseudodifferential operator by constant coefficient operators.

In addition to the conjectures that I enumerated above, I plan to study other properties of the pseudodifferential operators on fractals, such as the Gårding inequality and the L^p-boundedness of symbols in S^m_ρ. These projects are motivated by the desire to understand the wave and Schrödinger equation on fractals. Moreover, in a joint project with Robert Strichartz we want to use the techniques that we have developed for pseudodifferential operators in order to study the "harmonic extension property". To be more specific, suppose that h is a function that is harmonic on an open set U except at a point x. What conditions on h will guarantee that we can extend h to a harmonic function on U? We conjecture that h should be in a specific L^p space or a Sobolev space. If x is a so called junction point for the fractal, then the theory of distribution supported at a junction point that has been developed in [114] helps to provide an answer. For a generic point x the proof will be substantially harder.

Another application of our results that we want to investigate is the Poisson integration formula for product of fractals. Such a formula was provided in [134] under some very restrictive conditions. We conjecture that the Poisson formula holds as long as the heat kernel estimates (1) are satisfied.

2.6. **Exotic and forbidden symbols.** This project is a natural extension of the pseudodifferential operators project. Namely, following [126, Chapter VII], we want to consider the symbol class $S^m_{\rho,\delta}$ of smooth symbols $p : X \times \mathbb{R} \to \mathbb{C}$ that satisfy

$$\left| \left(\lambda^{-\delta}\Delta_x\right)^n \left(\lambda^{\rho}\frac{d}{d\lambda}\right)^k p(x,\lambda) \right| \leq C_{k,n}(\rho,\delta)(1+\lambda)^{\frac{m}{d+1}},$$

for all $n, m \geq 0$. Notice that for $\rho = 1$ and $\delta = 0$ we recover the pseudodifferential operators described above. If $\rho = 1$ and $\delta = 1$ one can not expect that the operator $Tu = p(x, -\Delta)$ is bounded on $L^2(\mu)$, as some well known examples in real analysis show. For this reason, we call the class $S^0_{1,1}$ the class of "forbidden" symbols (as in [126]). It would be interesting, however, to find an example of a symbol p defined on an infinite blow-up of a Sierpinski gasket such that $p(x, -\Delta)$ is not bounded on $L^2(\mu)$. Moreover, we plan to check if such operators are given by integration with respect to some kernels and to study the decay properties of these kernels. Also, we want to study the boundedness of these operators on different function spaces introduced in ([133]). In particular, we believe that the following is true.

Conjecture 2.10. *Suppose that the symbol p belongs to $S^0_{1,1}$. Then the operator $p(x, -\Delta)$ extend to a bounded operator on the Lipschitz space Λ_γ, for $\gamma > 0$.*

Next we want to study the class of "exotic" symbols, $S^0_{\rho,\rho}$ with $0 \leq \rho < 1$. By analogy with the real case, we want to prove the following conjecture:

Conjecture 2.11. *Suppose p is a symbol that belongs to $S^0_{\rho,\rho}$ with $0 \leq \rho < 1$. Then the operator $p(x, -\Delta)$ extends to a bounded operator on $L^2(\mu)$.*

The proof of this result for a p.c.f. will be very different compared to the proof of the similar result in the classical case. While I still expect that the Cotlar inequality ([22]) will play a crucial role in the proof, the symmetry between the x-space and λ-space that was essential in the real case fails here. The idea of proof that we want to pursue is to decompose the λ-space using the dyadic decomposition and to decompose the x-space using a particular cell decomposition of X. Once again, we have to be carefully on the x-space side because we can not multiply with smooth cut-off functions. The results in [115] should once more play an important role in overcoming these obstacles (see also [124],[127]). Success in proving these conjecture will shed more light about heat kernels on the imaginary axis and, perhaps, Cauchy-Szegö type kernels on fractals.

2.7. **Littlewood-Paley functions and area functions on fractals.** This project is another natural extension of the pseudodifferential operators on fractals project. Namely, we want to define and study the Littlewood-Paley functions and area function on fractals. While we can not define the so called g-function on fractals, it is easy to define the g_1-function

$$g_1(f)(x) = \left(\int_0^\infty te^{t\Delta}f(x)dt\right)^{1/2}.$$

A relatively easy proof shows that g_1 is bounded on $L^2(\mu)$. While using the general results of [125] one can prove that g_1 is bounded on $L^p(\mu)$, $1 < p < \infty$, it will be desirable to obtain a direct proof, using the self-similarity properties of the fractal X. A more delicate problem is the study of the so called area function. The definition of the area function on a generic infinite blowup of the Sierpinski gasket that we propose is the following. For a generic point $x \in X$ and any $m \in \mathbb{Z}$ there is a unique m-cell $C_m(x)$ containing x. We define

$$A(f)(x)^2 = \sum_{m=-\infty}^{\infty} \int_{3^{-m-1}}^{3^{-m}} E_{C_m(x)}(u(\cdot, t))dt, \tag{12}$$

where $u = e^{-t\sqrt{-\Delta}}f$ and E_C denotes the energy restricted to the cell C. We managed to prove the following results about the area function:

Theorem 2.12. *Suppose that A is defined as in* (12)*. Then $\|Af\|_2^2 \simeq \|f\|^2$. That is, there are positive constants C_1 and C_2 such that $\|A(f)\|_2 \leq C_1\|f\|_2$ and $\|f\|_2 \leq C_2\|A(f)\|_2$.*

The next step is to prove the following conjecture for the area function:

Conjecture 2.13. *The area function A is bounded on L^p and there positive constants $C_1(p)$ and $C_2(p)$ such that $\|A(f)\|_p \leq C_1(p)\|f\|_p$ and $\|f\|_p \leq C_2(p)\|A(f)\|_p$.*

Notice that this result can not be obtained using the general results of [125].

Another Littlewood-Paley function that we want to study is the partial sum operator. Strichartz defined in [135] a special case of the partial sum operator S on a fractal with *spectral gaps* by adding the Fourier coefficients only between gaps. Then $A_q\|f\|_q \leq \|Sf\|_q \leq B_q\|f\|_q$ for $1 < q < \infty$. We conjecture that the result is valid for fractals even in the absence of spectral gaps. We can define the sum operator in a similar fashion as in the Euclidean space using the dyadic decomposition ([124, Section 4.5.1]). Using our theory of pseudodifferential operators we already made progress in proving that $\|Sf\|_p \simeq \|f\|_p$ for all $1 < p < \infty$ for *all* p.c.f. self-similar sets as long as the heat kernel estimates are satisfied. The main motivation for the study of the partial sum operator S is our desire to understand Schrödinger type equations on fractals in a similar fashion as in [14]. In that paper the author studies the periodic nonlinear Schrödinger equation

$$\Delta_x u + i\partial_t u + u|u|^{p-2} = 0,$$

with $p \geq 3$, $u = u(x,t)$ a 1-periodic in each coordinate of the x-variable, and with some given initial data. We believe that understanding non-linear Schrödinger equations on fractals will be important to physicists. There are, however, a few problems that we need to be careful about; the first problem is how to pick the non-linear term. As explained above, the product of two smooth functions fails to be smooth on many highly-symmetric fractals. However, since the Schrödinger operator maps L^2 into L^2, it might suffice to assume that u is in L^2. Also, the basic arguments of [14] that use the algebraic properties of the exponential function (see, for example, the proof of [14, Proposition 2.1]) fail for general p.c.f. self-similar sets. However, some incipient computations lead us to believe that we can circumvent some of these difficulties using the properties of the sum operator on fractals. In some ongoing work done with Luke Rogers and Kasso Okoudjou, we realized that for fractals with localized eigenfunctions some of the results of [14] fail. Namely, the presence of the localized eigenfunctions implies that the Schrödinger operator $e^{it\Delta}$ can not be smoothing. That is, the operator can not be bounded from L^p to L^q if $q > p$. More generally, we conjecture that a multiplier must have a specific decay in order for the corresponding multiplier operator to be smoothing, decay that the Schrödinger multipliers fail to satisfy.

A closely related project that we believe will be of interest to people working in analysis on fractals and that we intend to pursue in the near future is the study of the weighted inequalities. Namely, we plan to define and study the A_p weights on fractals and study the connection between the weak (p,p) inequalities and A_p weights. This will most likely be a longer term project.

2.8. **Fredholm modules, Dirichlet forms, and derivations on fractals.** In [62] we study derivations and Fredholm modules on metric spaces with a local regular conservative Dirichlet form. The classical example of a Dirichlet form is $\mathcal{E}(u,u) = \int |\nabla u|^2$ with domain the Sobolev space of functions with one derivative in L^2. In [16] Cipriani and Sauvageot show that any sufficiently well-behaved Dirichlet form on a C^*-algebra has an analogous form, in that there is a map, δ, that is a derivation (i.e. has the Leibniz property) from the domain of the Dirichlet form to a Hilbert module $\mathcal{H}$, such that $\|\delta a\|^2 = \mathcal{E}(a,a)$. In the case that the Dirichlet form is local regular on a separable locally compact metric measure space, this construction is a variant of the energy measure construction of LeJan [81]. In particular, understanding the module $\mathcal{H}$ essentially relies on understanding energy measures. It is now well-known that fractal sets provide interesting examples of Dirichlet forms with properties different from those found on Euclidean spaces. Cipriani and Sauvageot study their derivation in the p.c.f. self-similar set setting in [17], obtaining properties of a Fredholm module (an abstract version of an order zero elliptic pseudodifferential operator in the sense of Atiyah [6]) using the heat kernel estimates and the counting function of the associated Laplacian spectrum. These results open up an exciting connection between Dirichlet forms on fractals and the non-commutative geometry of Connes [19], so it is natural to ask for an explicit description of the key elements of this connection, namely the Hilbert module $\mathcal{H}$ and its associated Fredholm module. In [62] we give a concrete description of the elements of the Hilbert module of Cipriani and Sauvageot in the setting of Kigami's resistance forms on finitely ramified fractals [75], a class which includes the p.c.f. self-similar sets studied in [17] and many other interesting examples [1] [139], [112]. We also discuss weakly summable Fredholm modules and the Dixmier trace in the cases of some finitely and infinitely ramified fractals (including non-self-similar fractals) if the so-called spectral dimension is less than 2. In the finitely ramified self-similar case we relate the p-summability question with estimates of the Lyapunov exponents for harmonic functions and the behavior of the pressure function. We give a direct sum decomposition of this module into piecewise harmonic components that correspond to the cellular structure of the fractal. This decomposition further separates the image of the derivation from its orthogonal complement and thereby gives an analogue of the Hodge decomposition for $\mathcal{H}$. By employing this decomposition to analyze the Fredholm module from [17] we give simpler proofs of the main results from that paper and further prove that summability of the Fredholm module is possible below the spectral dimension. We also clarify the connection between the topology and the Fredholm module by showing that there is a non-trivial Fredholm module if and only if the fractal is not a tree (i.e. not simply connected).

In the near future we want to extend this project into a few directions. A first natural avenue of research is the proof of the following conjecture:

Conjecture 2.14. *The Fredholm module of* [17] *and* [62] *is given by a pseudodifferential operator on the fractal in the sense of* [61].

Another natural direction given my background is to prove some of the results that I described above in the context of non-commutative Dirichlet forms ([2],[18],[16]). We already realized that the non-commutative version of the extension theorem [41, Theorem 1.5.2] that is essential in LeJan construction is the Stinespring's dilation theorem that is also one of the crucial tools used in [16]. In a different direction, we want to check whether the resolvent estimates that we proved in [59] might help in sharpening our results and extending them to other classes of fractals. Namely, do our results hold for diamonds and possible non-finitely ramified Laakso spaces ([79])? We conjecture that the answer is affirmative. How do our results change if the form is not local?

3. Fractals and C^*-algebras

3.1. **Disertation Results.** The results from [55], [54], and [63] have evolved from my dissertation. My thesis was completed at the University of Iowa under the direction of Professor Paul S. Muhly. It was awarded the 2006 D.C. Spriestersbach Dissertation Prize for Mathematics, Physical Sciences, and Engineering at the University of Iowa.

The common subject of the papers mentioned above is the structure of certain operator algebras (self-adjoint and non-self-adjoint) associated with Mauldin-Williams graphs and the dynamical systems they determine. My primary focus was the Pimsner construction of what are known now as Cuntz-Pimsner algebras (see [102] and [90]).

Let $G = (E^0, E^1, r, s)$ be a *finite* directed graph. A *Mauldin-Williams graph* associated to G consists of a collection $\{T_v\}_{v\in E^0}$ of compact metric spaces, one for each vertex of the graph, and a collection $\{\phi_e\}_{e\in E^1}$ of contractive maps, one for each edge of the graph ([86], [35]). We associate with such a system a C^*-correspondence $\mathcal{X}$ over the C^*-algebra $A = C(T)$, where $T = \bigsqcup T_v$, which reflects the dynamics of the Mauldin-Williams graph (see [55, Definition 2.2])). My interests lie in the structure of operator algebras built from this correspondence.

The first main result of [55] states that if the underlying graph G has no sources and no sinks, that is, if the maps r and s are surjective, then the Cuntz-Pimsner algebra $\mathcal{O}(\mathcal{X})$ associated to the Mauldin-Williams graph is isomorphic to the Cuntz-Krieger algebra associated with the graph G [23] (see [55, Theorem 2.3]). This is a generalization of the results of [103]. My proof is different from that in [103] and yields a second theorem. It asserts, roughly, that if one wants to build a graph-directed system where the T_v are replaced by arbitrary unital C^*-algebras A_v and where the ϕ_e are replaced by homomorphisms that are contractive in the Rieffel metric ([110],[111]) then the resulting Cuntz-Pimsner algebra *still* is isomorphic to $C^*(G)$. In fact, I proved that in such situations, using the hypothesis that the graph G has no sinks, the C^*-algebras A_v involved are necessarily commutative.

I showed, however, in [54] that the *tensor algebra* of $\mathcal{X}$, $\mathcal{T}_+(\mathcal{X})$, is "locally" a "complete conjugacy invariant". More precisely, I proved that if $\mathcal{X}_i$, $i = 1, 2$ are the C^*-correspondences coming from two Mauldin-Williams graphs defined over the same graph G, then the associated tensor algebras $\mathcal{T}_+(\mathcal{X}_i)$ are Morita equivalent in the sense of [12] if and only if that are completely isometrically isomorphic. This, in turn, happens if and only if there is a homeomorphism between the vertex spaces which implements a conjugacy between the appropriate edge maps. This result, thus, stands in a long series of results that were inspired by Arveson's discovery

[4] of the relation between conjugacy invariants for measure preserving transformations and non-self-adjoint operator algebras.

In the third paper that resulted from my thesis, [63], Watatani and I associate a *slightly* different C^*-correspondence to a Mauldin-Williams graph, which yields a C^*-algebra that seems to respect the dynamics more clearly. This new C^*-correspondence X over $A = C(K)$ is based on the union of the so called *cographs* of the maps ϕ_e. This approach allows us to put more emphasis on the "branch points" of the maps. These are points $(x, y) \in K \times K$ such that $\phi_e(y) = \phi_f(y) = x$ for some $e \neq f$. Assuming that the underlying graph G of a Mauldin-Williams graph is irreducible and is not a cyclic permutation, and assuming that the invariant set K satisfies a technical condition called the open set condition, we proved that the Cuntz-Pimsner algebra $\mathcal{O}(X)$ associated with the C^*-correspondence X is simple and purely infinite. Since $\mathcal{O}(X)$ is also separable, nuclear, and satisfies the Universal Coefficient Theorem, the classification theorem of Kirchberg and Phillips [77],[101] implies that the isomorphism class of $\mathcal{O}(X)$ is completely determined by its K-theory with the K-theory class of the unit. Its K-theory is closely related to the failure of the injectivity of the coding by the shift on a Cantor set. In particular, Watatani and I compute the K-theory for a few specific examples and show that it can be quite different from the K-theory of the underlying graph.

3.2. **Graph directed Markov systems and C^*-algebras.** This project is a natural extension of my thesis. A graph directed Markov system (GDMS) is a generalization of an Mauldin-Williams graph in that it allows for an infinite, but countable, number of edges, while still requiring a finite number of vertices [85]. To each vertex v one attaches a compact metric space K_v and to each edge e one attaches a contraction $\phi_e : K_{r(e)} \to K_{s(e)}$. Some examples of such dynamical systems are the so called continued fractions and the Kleinian groups of Schottky type. The main difference compared with the classical Mauldin-Williams graphs is that the invariant set and the path space of a GDMS fails, in general, to be a locally compact space. This failure makes it challenging to associate a C^*-algebra to a GDMS. The idea that I propose to overcome this difficulty is to use the groupoid model of Paterson [99] for infinite graph C^*-algebras. This leads to building a topological quiver over the closure of the invariant set. Natural questions I will like to answer include: how does the structure of the C^*-algebra associated with a graph directed Markov system depend on the underlying infinite graph? How is the dynamics of the GDMS reflected in the properties of the C^*-algebra? Is the associated C^*-algebra simple? Can one compute it's K-theory by studying particular sets of points? Based on the work I have done so far, the C^*-algebra associated to a GDMS should be, in general, different from the C^*-algebra of the underlying graph. Moreover, I have reasons to believe that these C^*-algebras are simple. This belief is substantiated by the work that I will describe in the next section. While I believe that, under suitable assumptions, these C^*-algebras are purely infinite, the proof seems to require more elaborate techniques compared with the classical Mauldin-Williams graphs.

In a different direction, I plan to study in collaboration with John Quigg and Steve Kaliszewski the KMS states induced from invariant measures of the GDMS on the C^*-algebra described above. This work will extend the impressive analysis of Pinzari, Watatani, and Yonetani [103].

3.3. **Markov operators and C^*-algebras.** In a recent paper which is joint with Paul S. Muhly and Victor Vega we began the study of Markov operators and C^*-algebras [58]. We say that an operator P on $C(X)$, where X is compact, is a Markov operator in case P is unital and positive. Using a Markov operator P, we built a topological quiver and a C^*-algebra $\mathcal{O}(P)$ ([92]) on $C(X)$ using the so called support of P. This Cuntz-Pimsner algebra, $\mathcal{O}(P)$ generalizes a number of C^*-algebras associated with automorphism, endomorphisms, transfer operators, and graphs ([123], [28], [29], [27], [38], [37], [73], [63], [104]). Our first theorem provides a characterization of the simplicity of the C^*-algebra $\mathcal{O}(P)$ in terms of the probabilistic properties of P. Namely, we proved that $\mathcal{O}(P)$ is simple if and only if there are no closed *strongly absorbing* sets for P. The second theorem of [58] states that, given a compact topological quiver E with no singular vertices, there is a Markov operator P such that $\mathcal{O}(E)$ is isomorphic to $\mathcal{O}(P)$. In future work I plan to explore further the interactions between the probabilistic features of P and the properties of $\mathcal{O}(P)$. One particularly intriguing problem is deciding when two P's give rise to isomorphic Cuntz-Pimsner algebras. When the P's are finite state Markov chains, then the two algebras are isomorphic if and only if the supports are same. Whether this is true more generally seems unlikely. However, [20, Proposition 6, p. 39] suggests that in general two P's with the same support give Morita equivalent Cuntz-Pimsner algebras.

Other interesting problem is to determine the "branch points" of a Markov operator and their influence on the K-theory of $\mathcal{O}(P)$. Namely, if $I_{\mathcal{X}} = C_0(U)$ is the ideal in $C(K)$ that is the preimage of the compact operators on $\mathcal{X}$ (see [102]), then we define the branch points of P to be the closed set $K \setminus U$. This definition generalizes work done in [73] and [72]. The existence of these branch points leads naturally to the study the KMS states on $\mathcal{O}(P)$ and representations of the C^*-algebra induced by invariant measures of the Markov operator P (see [69] for some particular cases). I expect that the ergodic Markov operators will provide interesting insights in the study of the KMS states on C^*-algebras. Representations of the Cuntz algebra $\mathcal{O}_n$ induced from invariant measures associated to iterated function systems have played a key role in the study of wavelet analysis done by Jorgensen, Bratelli, and co. [15], [32], [30], [70]. My work should unify and extend, thus, their study and provide new examples of wavelets on the real line and fractals.

3.4. **Tensor Algebras Associated to Fractals and their Perturbations.** As I pointed out in the description of my dissertation results, a natural C^*-correspondence associated with an iterated function system or, more generally, a Mauldin-Williams graph gives a C^*-algebra that ignores the dynamics of the system or graph. However, I proved in [54] that the tensor algebra does determine the dynamics in specified ways. One question to investigate is a perturbation question: If $\mathcal{X}_i$, is the C^*-correspondence coming from a Mauldin-Williams graph, $i = 1, 2$, and if the underlying graphs are the same, so that one may identify $\mathcal{O}(\mathcal{X}_1)$ and $\mathcal{O}(\mathcal{X}_2)$, under what circumstances is $\mathcal{T}_+(\mathcal{X}_1)$ close to $\mathcal{T}_+(\mathcal{X}_2)$ in the Hausdorff metric? I expect the answer to be in terms of some sort of "closeness" for the underlying dynamics. Another question I plan to pursue is whether the non-self adjoint algebras one can associate to the graph directed Markov systems I described above will provide a topological invariant for the GDMS in a similar way with [54]. Providing an answer for a GDMS should open a new set of problems and conjectures. For example, if the Toeplitz algebra is indeed a topological invariant of the system, for which Markov operators will the result still hold? Davidson and Katsoulis proved in [24] and [25] that the result would fail if one considers, in our language, a Markov operator built from a finite number of continuous maps that are *not* contractions.

4. Wavelets, Fractals, and Groupoids

4.1. **Groupoids methods in wavelet theory.** One project which I am actively pursuing with Paul S. Muhly is the use of groupoid methods in wavelet theory. An outline of our work together with partial results are published in [57]. We summarize the results below. The key idea is the use of the Renault-Deaconu groupoid [28], [108] and the theory of Exel [38] concerning irreversible dynamical systems to expand on the work of Bratteli, Jorgensen, Dutkay, et. al [15], [70], [31]. Their work, in turn, relates wavelet analysis, both for classical wavelets and for wavelets on fractals, to representations of the Cuntz algebra. Our approach shows how their Cuntz representations may be tied more closely to the underlying geometry of the situations they consider. A wavelet is a function ψ in $L^2(\mathbb{R})$ such that

$$\{U^j T^k \psi \ : \ j,k \in \mathbb{N}\}$$

is an orthonormal basis for $L^2(\mathbb{R})$, where U is the operator of dilation by 2, and T is the operator of translation by 1.

In our approach we start with a local homeomorphism T on a compact, Hausdorff space X, and define the *Renault-Deaconu groupoid* to be

$$G = \{(x,n,y) \in X \times \mathbb{Z} \times X \ : \ T^k(x) = T^l(y), n = k - l\}, \tag{13}$$

endowed with a suitable topology such that G becomes an étale, locally compact groupoid. Thus G carries information about the entire pseudogroup generated by T. In investigations that Muhly and I have been making (described in part in [57]) it has become clear that harmonic analysis on fractals and the analysis of wavelets can profitably exploit the representation theory of G for suitable choices of X and T. Our analysis shows that there are structures that are intrinsic to the geometric setting of a space X with a local homeomorphism T. These include the groupoid G and its C^*-algebra, the pseudogroup $\mathfrak{G}$, and the Deaconu correspondence $\mathcal{X}$ [28],[29]. These are the source of isometries and the Cuntz relations - assuming $\mathcal{X}$ has an orthonormal basis. Each choice of orthonormal basis (which we call a *filter bank*) gives Cuntz isometries S_i in $C^*(G)$. Further, we may construct the minimal unitary extension of any of the S_i *essentially within* $C^*(G)$.

The "classic" wavelet analysis arises from our groupoid perspective via the following example. Let $X = \mathbb{T}$, $Tz = z^2$, $\mu =$ Lebesgue measure on $\mathbb{T}$. Let π be the representation of $C^*(G)$ given by $(\mu, \mathcal{H}, U)$, where $\mathcal{H} = X \times \mathbb{C}$ is the trivial line bundle on X,

$$U(\gamma) : \{s(\gamma)\} \times \mathbb{C} \to \{r(\gamma)\} \times \mathbb{C},\ U(\gamma)(s(\gamma), c) = (r(\gamma), c).$$

This representation induces the classical wavelets: $\pi(f)\xi(z) = f(z)\xi(z)$, $\pi(S_i)\xi(z) = m_i(z)\xi(z^2)$, where (m_1, m_2) is a filter bank associated with T. Then one can recapture the result due to Jorgensen and, more recently, Larsen and Raeburn [80] that the inverse Fourier transform of $m_2(e^{\pi i x})\phi(2^{-1}x)$ is the wavelet associated with the filter bank (m_1, m_2).

4.2. **Local homeomorphisms on fractafolds.** Let T be a local homeomorphism on a compact metric space X. Alex Kumjian and I say in [56] that T satisfies the *local scaling condition* if

$$(x,y) \mapsto \frac{\rho(Tx,Ty)}{\rho(x,y)}$$

extends to a continuous function f on $X \times X$ that is strictly positive on the diagonal $\Delta_X = \{(x,x) \mid x \in X\}$. Our first theorem shows that the Hausdorff measure on X gives rise to a KMS state on the C^*-algebra attached to T via the Renault-Deaconu groupoid ([28, 108],[109]) with inverse temperature given by the Hausdorff dimension of X. Assuming that the local homeomorphism $T : X \to X$ is positively expansive and exact, and assuming that $\varphi = f|_\Delta$ satisfies the Bowen condition then we prove that β is the unique inverse temperature which admits a KMS state. Moreover, the (α,β)-KMS state ω_μ is unique. We use our results to derive a formula that allows an easy computation of the topological pressure of T in some particular cases. We discuss a few examples that generalize known results for gauge actions and generalized gauge actions on Cuntz and Cuntz-Krieger C^*-algebras. We present also the connection between the formula that computes the Hausdorff dimension of a self-similar set and the formula that defines the unique KMS-state attached to the gauge action on the Cuntz algebra. One of the main motivation for our work was an attempt to define and study a local homeomorphism on a fractafold based on a Sierpinski gasket. Since it seems unlikely that one can define a local homeomorphism on the Sierpinski gasket itself due to the different topological nature of the three vertex points compared with a "generic" point in the gasket, we looked at a simple fractafold that one can build using the Sierpinski gasket. Namely, we defined a natural local homeomorphism σ on the *Sierpinski octafold*, which is obtained by considering four copies of the Sierpinski gasket on alternating faces of a octahedron. We believe that our local homeomorphism is the fractal analogue of the map $T(z) = z^2$ on the torus. It turns out that this example did not satisfy the hypothesis of our main theorem in [56]. However, we showed using the associated Renault-Deaconu groupoid that the conclusion is still true. Moreover we computed the topological entropy of our local homeomorphism: $h(T) = \log 3$.

In recent and ongoing work done with Alex Kumjian, we try to find and analyze symmetries of fractals associated to iterated function systems $(F_1, \dots, F_N)$ and to study the associated C^*-algebras that might arise from the dynamics. Recall that in [131] Stricharz constructed a family of fractafold blowups of the invariant set of an iterated function system which is parameterized by infinite words in the alphabet $\{1, \dots, N\}$ and observed that two such blowups are naturally homeomorphic if the parametrizing words are eventually the same. We endow these fractafold blowups with the inductive limit topology and assemble them into a fractafold bundle L. In general there do not appear to be any natural nontrivial symmetries of a generic blowup but Stricharz's observation suggests that we look for symmetries of the bundle instead. Indeed we show that the homeomorphisms between fibers observed by Stricharz give rise to a natural action on L, the fractafold bundle, of the Cuntz groupoid. The groupoid action and the associated action groupoid $\tilde{G}$ constitute the main focus for our work. We prove that the there is a local homeomorphism $\tilde{\sigma}$ on L such that $\tilde{G}$ is isomorphic to the Renault-Deaconu groupoid associated to $\tilde{\sigma}$ and, in particular, $\tilde{G}$ is étale (that is, the range map is a local homeomorphism). We also prove that $\tilde{G}$ is topologically principal and has a dense orbit. If L is locally compact then we conjecture that the associated C^*-algebra, $C^*(\tilde{G})$, is primitive. Another conjecture that we make is that there is an invariant measure for $\tilde{G}$. As a consequence, the associated C^*-algebra would have a densely defined lower semi-continuous trace.

5. The structure of groupoid and Fell bundle C^*-algebras

5.1. **Induced representations and primitive ideals.** In this ongoing project, in which I am collaborating with Dana P. Williams, we are concerned with the generalization of the famous Effros-Hahn conjecture to groupoid and Fell bundle C^*-algebras. Key to this project is understanding the theory of representations of these C^*-algebras.

In two recent papers [65] and [64], Williams and I made significant progress on this project and we proved that a generalized Effros-Hahn conjecture is true for groupoid C^*-algebras. Let me begin with a review of the "classical" Effros-Hahn conjecture and a short description of our results. I will proceed, then, with a discussion of our future plans.

A dynamical system (A, G, α), where A is a C^*-algebra, G is a locally compact group and α is a strongly continuous homomorphism of G into $\mathrm{Aut}A$, is called *EH-regular* if every primitive ideal of the crossed product $A \rtimes_\alpha G$ is induced from a stability group ([144]). In their 1967 *Memoir* [36], Effros and Hahn conjectured that if (G, X) was a second countable locally compact transformation group with G amenable, then $\big(C_0(X), G, \mathrm{lt}\big)$ should be EH-regular. This conjecture, and its generalization to dynamical systems, was proved by Gootman and Rosenberg in [42] building on results due to Sauvageot [117],[118].

In [107], Renault gives the following version of the Gootman-Rosenberg-Sauvageot Theorem for groupoid dynamical systems. Let G be a locally compact groupoid and (A, G, α) a groupoid dynamical system. If R is a representation of the crossed product $A \rtimes_\alpha G$, then Renault forms the restriction, $\hat{L}$, of R to the isotropy groups of G and forms an induced representation $\mathrm{Ind}\,\hat{L}$ of $A \rtimes_\alpha G$ such that $\ker R \subset \ker(\mathrm{Ind}\,\hat{L})$. When G is suitably amenable, then the reverse conclusion holds. This is a powerful result and allows Renault to establish some very striking results concerning the ideal structure of crossed products.

In [64], Williams and I provide a significant sharpening of Renault's result in the case of a groupoid C^*–algebra — that is, a dynamical system where G acts on the commutative algebra $C_0(G^{(0)})$ by translation. We showed that every primitive ideal K of $C^*(G)$, with G amenable, is induced from a stability group. That is, we show that $K = \operatorname{Ind}_{G(u)}^G J$ for a primitive ideal J of $C^*\big(G(u)\big)$, where $G(u)$ is the stability group at some $u \in G^{(0)}$. This not only provides a cleaner generalization of the Gootman-Rosenberg-Sauvageot result to the groupoid setting, but gives us a much better means to study the fine ideal structure of groupoids and the primitive ideal space (together with its topology) in particular. We took the opportunity to formalize the theory of inducing representations from a general closed subgroupoid in [65]. The main result of this paper is that the induced representation of an irreducible representation of a stability group is irreducible. This result is also one of the main pillars in our proof of the Effros-Hahn conjecture for groupoids. In the case of transformation group C^*-algebras, it is well known that representations induced from irreducible representations of the stability groups are themselves irreducible [143]. The corresponding result for groupoid C^*-algebras has been proved in an *ad hoc* manner in a number of special cases (see, e.g., [105],[107],[89, 93, 94]). Thus our analysis unifies and extends these results to groupoid C^*-algebras.

Our next goal is to extend our results to other sorts of dynamical systems built on groupoids: twisted groupoids ([78]), Green twisted dynamical systems ([43]), groupoid dynamical systems and twisted groupoid dynamical systems ([107],[106]). Fortunately, as described in detail in [88, §3] or [96, §2], all these variants are subsumed using the C^*-algebra of a separable Fell bundle $p : \mathcal{B} \to G$ over a locally compact groupoid G with a Haar system. In this event, the sections $A = \Gamma_0(G^0, \mathcal{B})$ form a C^*-algebra and we prove in [67] that the groupoid G acts continuously on $\operatorname{Prim} A$. Any representation L of $C^*(G_P, \mathcal{B})$, where P is a primitive ideal in A, is associated to a representation π of the C^*-algebra A. We use the action defined in [67] to generalize a result about short exact sequences to Fell bundles over groupoids. Namely, we show that if I is a G-invariant ideal in A, then there is a short exact sequence of C^*-algebras

$$0 \longrightarrow C^*(G, \mathscr{B}_I) \longrightarrow C^*(G, \mathscr{B}) \longrightarrow C^*(G, \mathscr{B}^I) \longrightarrow 0,$$

where $C^*(G, \mathscr{B})$ is the Fell bundle C^*-algebra and $\mathscr{B}_I$ and $\mathscr{B}^I$ are naturally defined Fell bundles corresponding to I and A/I, respectively. Of course this exact sequence reduces to the usual one for C^*-dynamical systems. Our main result in [66] extends a classic Morita Equivalence result of Green's to the C^*-algebras of Fell bundles over groupoids. Specifically, we show that if $p : \mathscr{B} \to G$ is a saturated Fell bundle over a transitive groupoid G with stability group $H = G(u)$ at $u \in G^{(0)}$, then $C^*(G, \mathscr{B})$ is Morita equivalent to $C^*(H, \mathscr{C})$, where $\mathscr{C} = \mathscr{B}_H$. As an application, we show that if $p : \mathcal{B} \to G$ is a Fell bundle over a *group* G and if there is a continuous G-equivariant map $\sigma : \operatorname{Prim} A \to G/H$, where $A = B(e)$ is the C^*-algebra of $\mathcal{B}$ and H is a closed subgroup, then $C^*(G, \mathcal{B})$ is Morita equivalent to $C^*(H, \mathcal{C}^I)$ where $\mathcal{C}^I$ is a Fell bundle over H whose fibres are $A/I - A/I$-imprimitivity bimodule and $I = \bigcap\{P : \sigma(P) = eH\}$. Green's result is a special case of our application to bundles over groups.

Our main result in [68] says that if L is an irreducible representation of $C^*(G_P, \mathcal{B})$, $\ker \pi = P$ and π is *homogeneous*, then $\operatorname{Ind}_{G_P}^G L$ is irreducible. This result extends [33, Theorem 1.7] and some results of [117] and [118] and constitutes an important step towards the proof of the Effros-Hahn conjecture for Fell bundle C^*-algebras. We illustrate how it "trickles down" to other dynamical systems settings. Our proof requires an intermediate result which is of considerable interest on its own. Namely if $p : \mathcal{B} \to G$ is a separable Fell bundle over a locally compact groupoid G with Haar system, then we show that if $u \in G^0$, if $G(u)$is the stability group of u in G, and if L is an irreducible representation of $C^*(G(u), \mathcal{B})$, then $\mathrm{Ind}_{G(u)}^{G} L$ is an irreducible representation of $C^*(G, \mathcal{B})$.

We expect that the study of the Effros-Hahn conjecture for groupoid dynamical systems and Fell bundles to be substantially harder. The proof will require to retool the methods of Sauvageot [117], [118] and Gootman-Rosenberg [42] (see also [144]) to work in the context of Fell bundles. Our work in [68] already captures important results from [117] and [118]. Another significant observation that we made in [68] is that it suffices to consider Fell bundles over *groups* and, thus, we can work in a more familiar framework. Success here should lead to an important improvement upon [107] and give more information about the structure of the primitive ideal space and simplicity of Fell bundle C^*-algebras. In particular, we conjecture that if $p : \mathcal{B} \to G$ is a separable Fell bundle over a Hausdorff groupoid G such that the action of G on Prim is minimal and there exists a point in $\mathrm{Prim}\, A$ with discretely trivial isotropy, then the reduced C^*-algebra of the Fell bundle is simple. I will continue to work on this project with Dana P. Williams.

5.2. **Groupoids and Markov operators.** I plan to use the techniques developed in the previous project to study the problem of deciding for which Markov operators P there is a groupoid such that $\mathcal{O}(P)$ is isomorphic to the C^*-algebra of the groupoid. This problem was suggested to me by Jean Renault and it should fill an important gap in the literature of topological quivers [91] and C^*-algebras. Based on preliminary work, it seems that one needs some kind of "locally finiteness" assumption for the Markov operator P. An answer to this problem might shed some light on a more general problem: given a topological quiver $E = (E^0, E^1, r, s, \lambda)$, is there a groupoid G so that $\mathcal{O}(E)$ is isomorphic to $C^*(G)$? The answer is known to be true when r is a local homeomorphism, that is, when E is a topological graph in the sense of Katsura ([74]). An answer for general topological quiver has been searched by many people working in C^*-algebras. I believe that my study of Markov operators might help provide a (negative) answer to this open question.

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