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# MARKOV OPERATORS AND $C^*$-ALGEBRAS

MARIUS IONESCU, PAUL S. MUHLY, AND VICTOR VEGA

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Abstract. A Markov operator $P$ acting on $C(X)$, where $X$ is compact, gives rise to a natural topological quiver. We use the theory of such quivers to attach a $C^*$-algebra to $P$ in a fashion that reflects some of the probabilistic properties of $P$.

## 1. Introduction

Our objective in this note is to use the theory of topological quivers [30, 31] to study natural $C^*$-algebras that can be associated to Markov operators. In particular, we shall use the theory developed in [31] to decide when these $C^*$-algebras are simple. In addition, we will explore a number of examples that help illustrate how our analysis may be applied and we shall explore connections between Markov operators and topological quivers.

The term "Markov operator" is used in a variety senses in the probability literature. We adopt the following definition here and will discuss the terminology more in Remark 2.1.

**Definition 1.** Let $X$ be a compact Hausdorff space and let $C(X)$ denote the space of continuous, complex-valued functions on $X$. A *Markov operator* on $C(X)$ is a unital positive linear map $P$ on $C(X)$.

**Definition 2.** A *topological quiver* is a quintet $E = (E^0, E^1, r, s, \lambda)$, where $E^0$ and $E^1$ are second countable, locally compact Hausdorff spaces, $r$ and $s$ are continuous maps from $E^1$ onto $E^0$, with $r$ open, and where $\lambda = \{\lambda_v\}_{v\in E^0}$ is a family of measures on $E^1$such that the (closed) support of $\lambda_v$, $\operatorname{supp}\lambda_v$, equals $r^{-1}(v)$, and such that for each function $f \in C_c(E^1)$, the function $v \to \int_{E^1} f(x)\, d\lambda_v$

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lies in $C_c(E^0)$. The space $E^0$ is called the space of *vertices* of $E$, $E^1$ is the space of *edges*, $r$ and $s$ are called the *range* and *source* maps, respectively, and $\lambda$ is called the family of weights.

In a fashion that will be spelled out in a bit more detail in the next section, each Markov operator $P$ on $C(X)$ gives rise to a topological quiver $E$. The vertex space $E^0$ is $X$, the edge space $E^1$ is the "support" of $P$, a subspace of $X \times X$, the range and source maps are the left and right projections, respectively, and the family of weights is given by a continuous family of probability measures naturally associated to $P$. The $C^*$-algebra that we associate to $P$ and will denote by $\mathcal{O}(P)$ is the $C^*$-algebra of this quiver.

In the next section, we provide additional definitions and detail, and we provide a variety of examples (not exhaustive) to which our analysis applies. Section 3 is devoted to determining when $\mathcal{O}(P)$ is simple. Section 4 is devoted to applying our simplicity criteria to the examples described in Section 2. Finally, in Section 5 we address the question: Given a topological quiver $E = (E^0, E^1, r, s, \lambda)$ when can one find a Markov operator $P$ so that the $C^*$-algebra of $E = (E^0, E^1, r, s, \lambda)$ is isomorphic to $\mathcal{O}(P)$?

## 2. Definitions and Examples

Throughout this note, $X$ will be a fixed compact Hausdorff space, which we shall assume to be second countable. Also, $P$ will be a fixed Markov operator on $C(X)$. The Riesz representation theorem gives a continuous family of measures on $X$ and indexed by $X$, which we will write $p(\cdot, y)$, such that

$$(Pf)(y) = \int f(x)p(dx, y). \tag{1}$$

The fact that $P$ is unital implies that each $p(\cdot, y)$ is a probability measure. Further, we can always extend $P$ to the bounded Borel functions on $X$ via the formula $Pf(y) := \int f(x)p(dx, y)$ and, consequently, $p(U, y) = \int 1_U(x)p(dx, y) = P(1_U)(y)$ for all Borel sets $U \subseteq X$ and all $y \in X$. The natural topological quiver to associate to $P$ is the one that gives the so-called GNS correspondence for $P$. Our first objective in this section is to give details to support this assertion.

*Remark* 2.1. Before continuing we want to explain why some may find our formula for $P$ unconventional. At one point early in the theory, Markov operators were defined as certain operators acting on measures on measurable spaces. Various hypotheses were imposed to insure that the operators had "adjoints" that acted on the space of measurable functions. The formula for the adjoint action on functions was written

$$(Pf)(x) = \int p(x, dy) f(y). \tag{2}$$

Evidently, (2) is a transposed version of (1). We have chosen our notation, (1), to conform with the conventions from graph algebra theory and other situations where one builds operator algebras from not-necessarily-reversable dynamical systems. As a result, some of our formulas are transposed versions of formulas in the literature. We note, too, that in the probability literature, what we are calling a Markov operator is sometimes called a Markov-*Feller* operator. Feller identified conditions that insure that $P$, defined on measurable functions as in (1) or (2), leaves the space of continuous functions invariant (assuming, of course, the measure space is built on some topological space.) For a contemporary view of these issues, and references see [35].

It is well-known that since $C(X)$ is commutative, $P$ is completely positive. Consequently, the following definition makes sense.

**Definition 3.** The *GNS-correspondence* for $P$ (over the $C^*$-algebra $C(X)$), is the space $C(X) \otimes_P C(X)$, which is the separated completion of the algebraic tensor product $C(X) \odot C(X)$ in the pre-inner product defined by the formula $\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle := \overline{\eta_1} P(\overline{\xi_1} \xi_2) \eta_2$, and is endowed with the bimodule structure over $C(X)$ defined by the formula $a \cdot (\xi \otimes \eta) \cdot b := (a\xi) \otimes (\eta b)$.

**Definition 4.** The *support* of $P$ (or of $p$), denoted $\operatorname{supp}(P)$ (resp. $\operatorname{supp}(p)$) is the complement of the set of all points $(x, y) \in X \times X$ with the property that there is a neighborhood $U$ of $x$ such that the function $y \to p(U, y)$ vanishes in some neighborhood of $y$. Equivalently, $(x_0, y_0) \notin \operatorname{supp} P$ if and only if there is a neighborhood of $(x_0, y_0)$ of the form $U \times V$ such that $p(U, y) = 0$, for all $y \in V$.

**Proposition 2.2.** *Let $E^0$ be $X$, let $E^1 \subseteq X \times X$ be $\operatorname{supp}(P)$, define $r$ and $s$ by the formulae $s(x, y) = x$ and $r(x, y) = y$, respectively, and define $\lambda_v := p(\cdot, v)$, $v \in E^0$. Then the quintet $E = (E^0, E^1, r, s, \lambda)$ is a topological quiver, with $E^1$ compact. Further, $C(E^1)$ becomes a $C^*$-correspondence $\mathcal{X}$ over $C(X)$ via the formula*

$$a \cdot \xi \cdot b(x, y) = a(x) \xi(x, y) b(y),$$

*and*

$$\langle \xi, \eta \rangle (y) = \int \overline{\xi(x, y)} \eta(x, y) p(dx, y) = \int \overline{\xi} \eta \, d\lambda_y,$$

$\xi, \eta \in C(E^1)$, $a, b \in C(E^0) = C(X)$, *and the map* $W : C(X) \otimes_P C(X) \to \mathcal{X}$ *defined by* $W(f \otimes g)(x, y) = f(x)g(y)$ *extends to an isomorphism of correspondences from the GNS-correspondence for* $P$ *to* $\mathcal{X}$.

Proof. With all the definitions before us, the proof is nothing but a straightforward process of checking. Of course $E^1$ is compact, since it is a closed subset of $X \times X$. The other matters are equally easy. □

The correspondence $\mathcal{X}$ is the correspondence of the kind that is associated to any topological quiver [31, Subsection 3.1].

**Definition 5.** The topological quiver associated to $P$ in Proposition 2.2 will be denoted $E(P)$ and the resulting correspondence will be denoted $\mathcal{X}(P)$.

*Remark* 2.3. The topological quiver $E(P)$ is a topological relation in the sense of Brenken [7] since $E^1$ is a closed subset of $X \times X$.

To define the $C^*$-algebra that we associate to $P$, and to relate it to $E(P)$, it will be helpful to spell out additional definitions and facts that will also play a role elsewhere in this note. Given a $C^*$-correspondence $\mathcal{X}$ over a $C^*$-algebra $A$ a *Toeplitz representation* of $\mathcal{X}$ in a $C^*$-algebra $B$ consists of a pair $(\psi, \pi)$, where $\psi : \mathcal{X} \to B$ is a linear map and $\pi : A \to B$ is a $*$-homomorphism such that

$$\psi(x \cdot a) = \psi(x)\pi(a) \ , \ \psi(a \cdot x) = \pi(a)\psi(x),$$

i.e. the pair $(\psi, \pi)$ is a *bimodule map*, and such that

$$\psi(x)^*\psi(y) = \pi(\langle x, y\rangle_A).$$

That is, the map $\psi$ preserves inner products (see [19, Section 1]). Given such a Toeplitz representation, there is a $*$-homomorphism $\pi^{(1)}$ from $\mathcal{K}(\mathcal{X})$ into $B$ which satisfies

$$\pi^{(1)}(\Theta_{x,y}) = \psi(x)\psi(y)^* \text{ for all } x, y \in \mathcal{X}, \tag{3}$$

where $\Theta_{x,y} = x \otimes \tilde{y}$ is the rank one operator defined by $\Theta_{x,y}(z) = x \cdot \langle y, z\rangle_A$. The $*$-homomorphism extends naturally to $\mathcal{L}(\mathcal{X})$ by virtue of the fact that $\mathcal{L}(\mathcal{X})$ is the multiplier algebra of $\mathcal{K}(\mathcal{X})$ and we will denote the extension by $\pi^{(1)}$ also. This extension $\pi^{(1)}$ and $\psi$ are related by the useful formula

$$\pi^{(1)}(T)\psi(\xi) = \psi(T\xi), \tag{4}$$

$T \in \mathcal{L}(\mathcal{X})$ and $\xi \in \mathcal{X}$. Indeed, if $T$ is a rank one operator, $\Theta_{x,y}$, then

$$\pi^{(1)}(T)\psi(\xi) = \psi(x)\psi(y)^*\psi(\xi) = \psi(x)\pi(\langle y, \xi\rangle) = \psi(\Theta_{x,y}(\xi)).$$

So the formula holds for all $T \in \mathcal{K}(\mathcal{X})$. If $T \in \mathcal{L}(\mathcal{X})$ and $S \in \mathcal{K}(\mathcal{X})$, then

$$\begin{aligned}\pi^{(1)}(T)\psi(S\xi) &= \pi^{(1)}(T)\pi^{(1)}(S)\psi(\xi) = \pi^{(1)}(TS)\psi(\xi)\\ &= \psi((TS)\xi) = \psi(T(S\xi)),\end{aligned}$$

since $\mathcal{K}(\mathcal{X})$ is an ideal in $\mathcal{L}(\mathcal{X})$ and $\mathcal{L}(\mathcal{X})$ is the multiplier algebra of $\mathcal{K}(\mathcal{X})$. Thus $\pi^{(1)}(T)\psi(\xi) = \psi(T\xi)$ for all $T \in \mathcal{L}(\mathcal{X})$ and $\xi \in \mathcal{X}$.

Let $\Phi : A \to \mathcal{L}(\mathcal{X})$ be the $*$-homomorphism that defines the left action of $A$ on $\mathcal{X}$. We define then

$$J(\mathcal{X}) := \Phi^{-1}(\mathcal{K}(\mathcal{X})),$$

which is a closed two sided-ideal in $A$ (see [19, Definition 1.1]) and we set $J_{\mathcal{X}} := J(\mathcal{X}) \bigcap \ker^{\perp}\Phi$, where $\ker^{\perp}\Phi$ denotes the set of all $a \in A$ such that $ab = 0$ for all $b \in \ker\Phi$. Suppose $K$ is any ideal in $J(\mathcal{X})$. We say that a Toeplitz representation $(\psi, \pi)$ of $\mathcal{X}$ is *coisometric on $K$* if

$$\pi^{(1)}(\Phi(a)) = \pi(a) \text{ for all } a \in K.$$

When $(\psi, \pi)$ is coisometric on all of $J(\mathcal{X})$, we say that it is *Cuntz-Pimsner covariant.*

It is shown in [19, Proposition 1.3] that for an ideal $K$ in $J(\mathcal{X})$, there is a $C^*$-algebra $\mathcal{O}(K, \mathcal{X})$, called the *relative Cuntz Pimsner algebra associated to $\mathcal{X}$ and* $K$, and a Toeplitz representation $(k_{\mathcal{X}}, k_A)$ of $\mathcal{X}$ into $\mathcal{O}(K, \mathcal{X})$, which is coisometric on $K$, and satisfies:

(1) for every Toeplitz representation $(\psi, \pi)$ of $\mathcal{X}$ which is coisometric on $K$, there is a $*$-homomorphism $\psi \times_K \pi$ of $\mathcal{O}(K, \mathcal{X})$ such that $(\psi \times_K \pi) \circ k_{\mathcal{X}} = \psi$ and $(\psi \times_K \pi) \circ k_A = \pi$; and
(2) $\mathcal{O}(K, \mathcal{X})$ is generated as a $C^*$-algebra by $k_{\mathcal{X}}(\mathcal{X}) \cup k_A(A)$.

The algebra $\mathcal{O}(\{0\}, \mathcal{X})$, then, is the *Toeplitz algebra $\mathcal{T}_{\mathcal{X}}$*, and $\mathcal{O}(J_{\mathcal{X}}, \mathcal{X})$ is defined to be the *Cuntz-Pimsner algebra $\mathcal{O}_{\mathcal{X}}$* (see [25] and [31]).

The parts of the following definition are taken from [31, Section 3]

**Definition 6.** Let $E = (E^0, E^1, r, s, \lambda)$ be a topological quiver and let $\mathcal{X}$ be the $C^*$-correspondence over $A = C(E^0)$ associated to $E$.

(1) The set of *sinks* of $E$, $E^0_{sinks}$, is defined to be the open subset $U$ of $E^0$ that supports the kernel of $\Phi$, i.e., $U$ satisfies the equation $\Phi^{-1}(0) = C_0(U)$.
(2) The set of *finite emitters* of $E$, $E^0_{fin}$, is defined to be the open subset of $E^0$ that supports $\Phi^{-1}(\mathcal{K}(\mathcal{X}))$, i.e., $\Phi^{-1}(\mathcal{K}(\mathcal{X})) = C_0(E^0_{fin})$.
(3) A vertex $v$ is called regular if it is a finite emitter, but not a sink. The set of all regular vertices is denoted $E_{reg}$, so that $E^0_{reg} = E^0_{fin} \setminus E^0_{sinks}$.

(4) Elements of $E^0 \setminus E^0_{fin}$ are called *infinite emitters.*
(5) The $C^*$-algebra $C^*(E)$ associated to $E$ is defined to be the relative Cuntz-Pimsner algebra $\mathcal{O}(C_0(E^0_{reg}), \mathcal{X})$.

*Remark* 2.4. Since the Markov operator $P$ is unital, $E(P)$ has no *sources*, that is $r(E^1) = E^0$. Otherwise, if $x \in E^0 \setminus r(E^1)$ then $P(1)(x) = 0$, by a compactness argument, which is a contradiction. Further, since $s(E^1)$ is compact, and hence closed, $E_{sinks} = E^0 \setminus s(E^1)$. Moreover $x$ is a sink if and only if there is an open neighborhood $U$ of $x$ such that $P(1_U)(y) = 0$ for all $y \in E^0$. Indeed, if $x$ is sink then for each $y \in E^0$ there is a neighborhood $U_y$ of $x$ and a neighborhood $V_y$ of $y$ such that $z \mapsto p(U_y, z)$ is 0 on $V_y$. Since $E^0$ is compact, there is a finite subcover $\{V_1, V_2, \dots, V_n\}$ of $X$. Let $U = \bigcap_{i=1}^n U_i$, which is open. It follows that $z \mapsto p(U, z) = 0$ for all $z \in E^0$, that is, $P(1_U) = 0$. The converse is clear. Finally, note that because $P$ is unital, $\mathcal{X}(P)$ is *full*, meaning that the ideal in $C(X)$ generated by the inner products equals $C(X)$.

**Definition 7.** The $C^*$-algebra of $P$ is defined to be $C^*(E(P))$ and will be denoted $\mathcal{O}(P)$.

**Example 2.5.** If $X = \{x_1, \dots, x_n\}$ is a finite set, then $P$ is a Markov operator on $X$ if and only if there is a stochastic matrix $p$ such that

$$P(f)(x_j) = \sum_i f(x_i) p_{ij},$$

for all $f \in C(X)$. To say $p$ is stochastic means here that $\sum_i p_{ij} = 1$, which is the transpose of the usual definition (see, for example [28]). In this case, of course, $E(P)$ is a finite directed graph with vertices $E^0 = X$, and the set of edges consists of the pairs $(x_i, x_j)$ such that $p_{ij} > 0$. Consequently, $\mathcal{O}(P)$ is the graph $C^*$-algebra studied by the third author in [34]. More accurately, he initially defined $\mathcal{O}(P)$ to be the Cuntz-Pimsner algebra of the $GNS$-correspondence determined by $P$ and arrived at the representation of $\mathcal{O}(P)$ as a graph $C^*$-algebra through (a special case of) Proposition 2.2.

**Example 2.6.** If $\tau : X \to X$ is a homeomorphism, and if $p(\cdot, y)$ is the point mass at $\tau^{-1}(y)$, then $P$ is the automorphism $\alpha$ of $C(X)$ given by the formula $\alpha(f)(y) = f(\tau^{-1}(y))$ since

$$Pf(y) = \int f(x)p(dx, y) = \int \delta_{\tau^{-1}(y)}(dx) f(x) = f(\tau^{-1}(y)) = \alpha(f)(y).$$

The support of $P$ is the graph of $\tau$. Thus the topological quiver is $E(P) = (E^0, E^1, r, s, \lambda)$ where $E^0 = X$, $E^1$ is the graph of $\tau$, i.e. $E^1 = \{(x, \tau(x)) : x \in$

$X\}$, $r(x,\tau(x)) = \tau(x)$, $s(x,\tau(x)) = x$, and

$$\begin{aligned} a \cdot \xi \cdot b(x,\tau(x)) &= a(x)\xi(x,\tau(x))b(\tau(x)), \\ \langle \xi, \eta \rangle(y) &= \overline{\xi(\tau^{-1}(y), y)}\eta(\tau^{-1}(y), y). \end{aligned}$$

The $C^*$-algebra $\mathcal{O}(P)$ is then the cross-product $C^*$-algebra $C(X) \rtimes_\alpha \mathbb{Z}$. Of course, viewing crossed products as Cuntz-Pimsner algebras was one of Pimsner's sources of inspiration [32].

**Example 2.7.** More generally, let $\tau : X \to X$ be a local homeomorphism. Then $\tau^{-1}(y)$ is a finite set for all $y \in X$. If $p(\cdot, y)$ is counting measure on $\tau^{-1}(y)$ normalized to have total mass 1, then

$$P(f)(y) = \frac{1}{|\tau^{-1}(y)|} \sum_{x \in \tau^{-1}(y)} f(x).$$

The support of $P$ is still the graph of $\tau$. The only difference from the previous example is the formula for the inner product, which becomes

$$\langle \xi, \eta \rangle(y) = \frac{1}{|\tau^{-1}(y)|} \sum_{x \in \tau^{-1}(y)} \overline{\xi(x,y)}\eta(x,y).$$

The $C^*$-algebra $\mathcal{O}(P)$ is the cross-product of $C(X)$ by the local homeomorphism $\tau$ studied extensively in [18, 8, 17, 16, 1, 10, 11, 12, 13, 9, 29, 23, 15].

**Example 2.8.** Markov operators have played a role in the theory of iterated function systems from the very begining of the theory fractals. See Hutchinson's paper [21] where they are mentioned explicitly in this context. As Hutchinson notes, many of the ideas he develops can be traced back further in geometric measure theory. Barnsley and his collaborators used Markov operators to good effect in encoding and decoding pictures in terms of fractals. For a sampling of this literature, see [3, 5, 4]. If $X$ is a compact metric space, an *iterated function system* on $X$ is a finite set of injective contractions $(f_1, f_2, \ldots, f_N)$, $N \geq 2$, on $X$. Given such an iterated function system there is a unique compact subset $K$ of $X$ which is invariant for the iterated function system, that is

$$K = f_1(K) \bigcup f_2(K) \bigcup \cdots \bigcup f_n(K).$$

In the following we assume that $X = K$. Then

$$P(f)(y) := \frac{1}{N} \sum_{i=1}^{N} f \circ f_i(y)$$

is a Markov operator on $C(X)$.

We claim that the support of $P$ equals $\bigcup_{i=1}^N \operatorname{cograph} f_i$, where for a function $f : X \to X$ the *cograph* of $f$ is

$$\operatorname{cograph} f = \{(x, y) : x = f(y)\}.$$

Indeed, if $(x, y)$ belongs to the support of $P$ then for any neighborhoods $U$ and $V$ of $x$ and $y$, respectively, there exists $z \in V$ and $i \in \{1, \ldots, N\}$ such that $f_i(z) \in U$. Therefore we can find a sequence $\{z_n\}_n$ that converges to $y$ and a sequence of indices $\{i_n\}_n \subset \{1, \ldots, N\}$ such that $\lim_{n\to\infty} f_{i_n}(z_n) = x$. There must be an index $i \in \{1, \ldots, N\}$ so that $i_n = i$ infinitely many times. Therefore there is a subsequence $\{z_{n_k}\}$ so that $\lim_{k\to\infty} f_i(z_{n_k}) = x$. Thus $x = f_i(y)$ and $(x, y)$ belongs to the cograph of $f_i$. The converse inclusion is clear. Thus the topological quiver $E(P)$ is given by $E^0 = X$, $E^1 = \bigcup_{i=1}^N \operatorname{cograph} f_i$, $s(x, y) = x$, $r(x, y) = y$. The actions and inner product on $\mathcal{X}(P)$ are given by

$$\begin{aligned} (a \cdot \xi \cdot b)(x, y) &= a(x)\xi(x, y)b(y) \\ \langle \xi, \eta \rangle(y) &= \frac{1}{N} \sum_{i=1}^N \overline{\xi(f_i(y), y)}\eta(f_i(y), y). \end{aligned}$$

Then the $C^*$-algebra $\mathcal{O}(P)$ is the $C^*$-algebra studied in [24] and in [23].

More generally, if $p = \{p_1, p_2, \ldots, p_N\}$ are probabilities ($p_i > 0$ for all $i$ and $\sum_{i=1}^N p_i = 1$), then $P_p(f)(y) := \sum_{i=1}^N p_i f \circ f_i(y)$ is a Markov operator on $C(X)$. Since $p_i > 0$ for all $i \in \{1, \ldots, N\}$ it follows that the support of $P_p$ is still $\bigcup_{i=1}^N \operatorname{cograph} f_i$. The only difference between $\mathcal{X}(P)$ and the $C^*$-correspondence $\mathcal{X}(P_p)$ associated to $P_p$ is the formula for the inner product, which is

$$\langle \xi, \eta \rangle_p(y) = \sum_{i=1}^N p_i \overline{\xi(f_i(y), y)}\eta(f_i(y), y).$$

We claim that $\mathcal{X}(P)$ and $\mathcal{X}(P_p)$ are isomorphic $C^*$-correspondences ([30]). To prove the claim recall first that for $(x, y)$ in the support of $P$ (which is the same as the support of $P_p$) we define it's branch index to be $e(x, y) = \#\{i \in \{1, \ldots, N\} : f_i(y) = x\}$ (see [24] and [23]). Then one can prove that the map $\psi : \mathcal{X}(P) \to \mathcal{X}(P_p)$ defined by

$$\psi(\xi)(x, y) = \frac{e(x, y)^{1/2}}{\sqrt{N}\left(\sum_{i\,:\,f_i(y)=x} p_i\right)^{1/2}} \xi(x, y),$$

for $\xi \in C(E^1)$, is a $C^*$-correspondence isomorphism. Thus $\mathcal{O}(P)$ and $\mathcal{O}(P_p)$ are isomorphic as $C^*$-algebras. For simplicity we will assume in the sequel that $p_1 = p_2 = \cdots = p_N = 1/N$.

**Example 2.9.** Let $X = [0,1]$, let $f_1(x) = x$, and let $f_2(x) = 1 - x$. Then $f_1$ and $f_2$ are not contractions, so $(f_1, f_2)$ is not an iterated function system. Nevertheless,

$$P(f)(y) := \frac{1}{2} \sum_{i=1}^{2} f \circ f_i(y)$$

is a Markov operator. More generally, if $(f_1, f_2, \ldots, f_N)$ are continuous functions on $X$ so that $X = f_1(X) \bigcup f_2(X) \bigcup \cdots \bigcup f_N(X)$, then

$$P(f)(y) := \frac{1}{N} \sum_{i=1}^{N} f \circ f_i(y)$$

is a Markov operator on $C(X)$. The support of $P$ is still $\bigcup_{i=1}^{N}$ cograph $f_i$.

**Example 2.10.** If $X = [0,1]$ and $p(\cdot, y) = m$, Lebesgue measure, for all $y \in X$, then $P$ is given by the formula

$$P(f)(y) := \int_{[0,1]} f(x) dm(x)$$

for all $y \in X$. The support of $P$ is $X \times X$, $E^0 = X$, $E^1 = X \times X$, $r(x,y) = y$, $s(x,y) = x$, $\lambda_y = m$ for all $y \in X$. The actions and inner product on the associated $C^*$-correspondence are given by the formulae

$$\begin{aligned} a \cdot \xi \cdot b(x,y) &= a(x)\xi(x,y)b(y), \\ \langle \xi, \eta \rangle (y) &= \int \overline{\xi(x,y)} \eta(x,y) dm(x). \end{aligned}$$

This is the prototypical example of a topological quiver for which every vertex is an infinite emitter.

## 3. Simplicity of the $C^*$-algebras

Given a topological quiver $E = (E^0, E^1, r, s, \lambda)$, the $C^*$-algebra $C^*(E)$ is simple if and only if $E$ satisfies condition (L) and the only open saturated hereditary subsets of $E^0$ are $E^0$ and $\emptyset$ [31, Theorem 10.2]. In this section we investigate these conditions for the topological quivers derived from Markov operators. For this we need to review a few definitions (see, for example [31, 26, 27]).

A *path of length* $n$ in a topological quiver $E = (E^0, E^1, r, s, \lambda)$ is a finite sequence $\alpha = \alpha_1 \alpha_2 \ldots \alpha_n$ so that $\alpha_i \in E^1$ and $r(\alpha_i) = s(\alpha_{i+1})$ for all $i = 1, \ldots, n-1$. We denote by $E^n$ the set of paths of length $n$, we write $E^* := \bigcup_{n \geq 0} E^n$ for the set of all *finite* paths, and we write

$$E^\infty := \{\alpha = (\alpha_n)_{n \geq 0} \,:\, \alpha_n \in E^1 \text{ and } r(\alpha_n) = s(\alpha_{n+1})\}$$

for the set of *infinite paths*. For a finite path $\alpha = \alpha_1\alpha_2\dots\alpha_n$ we define $s(\alpha) := s(\alpha_1)$ and $r(\alpha) := r(\alpha_n)$. If $\alpha$ is an infinite path we define $s(\alpha) := s(\alpha_1)$. A finite path $\alpha$ is called a *loop* if $r(\alpha) = s(\alpha)$. In this case we say that $v = r(\alpha) = s(\alpha)$ is a *base* of the loop $\alpha$. If $\alpha = \alpha_1\alpha_2\dots\alpha_n$ is a loop, then an *exit for* $\alpha$ is an edge $\beta \in E^1$ such that $s(\beta) = s(\alpha_i)$ for some $i \in \{1, \dots, n\}$ and $\beta \neq \alpha_i$ ([31, Definition 6.8]). Thus, if $x = s(\alpha_i)$, there must be at least two edges leaving $x$, i.e., $|s^{-1}(x)| \geq 2$.

**Definition 8.** A topological quiver is said to satisfy *condition (L)* (in the sense of [31, Definition 6.9]) if the set of base points of loops with no exit has empty interior.

Turning now to condition (L) for our Markov operator $P$, observe that $P^n$ is also Markov operator on $C(X)$ for all $n \geq 1$. In the next lemma we relate the paths of $E$ to the powers of $P$.

**Proposition 3.1.** *For $v, w \in X$ there is a path of length $n$ from $v$ to $w$ if and only if for any neighborhood $V$ of $v$ and neighborhood $W$ of $w$ there is $f \in C(X)$ with* $\operatorname{supp} f \subset V$ *and $f \geq 0$, such that $P^n(f)(z) > 0$ for some $z \in W$.*

Proof. In one direction, we proceed by induction based on the length of the path from $v$ to $w$. If the length of the path is one, the condition means that $(v, w)$ belongs to the support of $P$. The conclusion is, then, the definition of the support of $P$. Assume next that there is a path of length two from $v$ to $w$, that is, assume there is an $\alpha = \alpha_1\alpha_2$, $\alpha_i \in E^1$, such that $s(\alpha) = v$, $r(\alpha) = w$. Let $V$ be an open neighborhood of $v$ ad $W$ an open neighborhood of $w$. Then for any open neighborhood $V_1$ of $r(\alpha_1) = s(\alpha_2)$ there are $f \in C(X)$ with $\operatorname{supp} f \subset V$ and $f \geq 0$, and $f_1 \in C(X)$ with $\operatorname{supp} f_1 \subset V_1$ and $0 \leq f_1 \leq 1$, such that $P(f_1 P(f))(z) > 0$ for some $z \in W$. Then

$$\begin{aligned} P^2(f)(z) &= \int_X P(f)(y)p(dy, z) \geq \int_X f_1(y)P(f)(y)p(dy, z) \\ &= P(f_1 P(f))(z) > 0. \end{aligned}$$

The inductive step is now clear. The other direction is immediate. □

Next we consider saturated hereditary sets for Markov operators basing the terminology on [31, Definition 8.3].

**Definition 9.** If $(E^0, E^1, r, s, \lambda)$ is topological quiver, then a subset $U \subseteq E^0$ is *hereditary* if whenever $\alpha \in E^1$ and $s(\alpha) \in U$, then $r(\alpha) \in U$. A subset $U$ of $X$ will be called hereditary for the Markov operator $P$ in case $U$ is hereditary for

$E(P)$. An hereditary subset $U$ of $E^0$ is called *saturated* if whenever $x \in E^0_{reg}$ and $r(s^{-1}(x)) \subseteq U$, then $x \in U$.

**Lemma 3.2.** *Suppose $y \in X$ and there is an open set $U$ such that $\int_U p(dx, y) > 0$. Then there is an $x \in U$ such that $(x, y)$ belongs to the support of $P$.*

PROOF. Since $X$ is assumed to be a second countable compact space, $X$ is metrizable. Therefore, we may choose a complete metric on $X$ so that the topology is given by the metric. Since $p(\cdot, y)$ is a Radon measure there is $x_1 \in U$ and $r_1 > 0$ so that if $V_1 = B(x_1, r_1)$ - the ball of radius $r_1$, centered at $x_1$, computed with respect to the metric - then $\int_{V_1} p(dx, y) > 0$ and $V_1 \subseteq \overline{V}_1 \subseteq U$. Inductively we may find a sequence of points $\{x_n\}$ and radii $\{r_n\}$ such that $r_n \to 0$ and if $V_n := B(x_n, r_n)$, then $V_n \supset V_{n+1}$ and $\int_{V_n} p(dx, y) > 0$ for all $n \geq 1$. The sequence $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy and so we may let $x := \lim_{n\to\infty} x_n$. Then from construction it follows that $x \in U$ and $\int_V p(dx, y) > 0$ for all neighborhoods $V$ of $x$. Thus $(x, y)$ belongs to the support of $P$. $\square$

**Proposition 3.3.** *If $U$ is an open subset of $X$ that is hereditary for the Markov operator $P$, then $P$ restricts to a Markov operator $P_K$ on $C(K)$, where $K = X \setminus U$.*

PROOF. Recall that the quotient $C(X)/C_0(U)$ is $*$-isomorphic with $C(K)$ via the map $[f] \to f|_K$, where $[f]$ is the equivalence class of an element $f \in C(X)$ in $C(X)/C_0(U)$, and $f|_K$ is the restriction of $f$ to $K$. Therefore it is enough to check that if $U$ is hereditary then $P$ maps $C_0(U)$ into $C_0(U)$. This is clear, however, since if $\int_U p(dx, y) > 0$ for some $y \in K$ it follows that that there is $x \in U$ such that $(x, y) \in E^1$ by Lemma 3.2. Then $y \in U$ since $U$ is assumed to be hereditary, and this is a contradiction. $\square$

Recall from [31, Definition 8.8] that if $\mathcal{E} = (E^0, E^1, r, s, \lambda)$ is a topological quiver and $U$ is a hereditary open subset of $E^0$, then one can "cut $\mathcal{E}$ down to $U$" to obtain a topological quiver $\mathcal{E}_U := (E^0_U, E^1_U, r_U, s_U, \lambda_U)$, where $E^0_U = E^0 \setminus U$, $E^1_U = E^1 \setminus r^{-1}(U)$, $r_U = r|_{E^1_U}$, $s_U = s|_{E^1_U}$, and $\lambda_U = \lambda|_{E^0_U}$. Thus, if $\mathcal{E}$ is the topological quiver associated with a Markov operator and $U$ is a hereditary set, then $\mathcal{E}_U$ is the topological quiver associated with $P_K$, where $K = X \setminus U$ and $P_K$ is the restriction of $P$ to $C(K)$. Recall also the following definition from the theory of Markov chains (see, e.g., [20, Definition 2.2.2]).

**Definition 10.** A Borel subset $B$ of $X$ is called *absorbing* with respect to the Markov operator $P$, if $p(B, x) = 1$ whenever $x$ lies in $B$.

Thus, $B$ is absorbing for $P$ if and only if $P(1_B)$ is identically 1 on $B$.

**Proposition 3.4.** *If $U$ is an open hereditary set, then $K := X \setminus U$ is a closed absorbing set.*

Proof. If $K$ is not an absorbing set, then there is a $y \in K$ such that $\int_U p(dx, y) > 0$. Lemma 3.2 implies that there is an $x \in U$ such that $(x, y)$ belongs to the support of $P$. Since $U$ is hereditary it follows that $y \in U$, which is a contradiction. □

**Proposition 3.5.** *A closed set $K$ is absorbing set for $P$ if and only if $P$ restricts to a Markov operator $P_K$ on $C(K)$.*

Proof. Suppose $K$ is an absorbing set for $P$. Then $P(1_K)(x) = 1$ for all $x \in K$. It follows that if $f \in C_0(U)$ then $P(f)(x) = 0$ for all $x \in K$. Thus $P(f) \in C_0(U)$. Hence $P$ restricts to a Markov operator on $C(K)$. Conversely, if $P$ restricts to a Markov operator $P_K$ on $C(K)$, then for any $f \in C(X)$, with $f|_K = 1$, we have $P_K([f])(x) = 1$ for all $x \in K$. Thus $p(K, x) = 1$ for all $x \in K$, and $K$ is absorbing for $P$. □

**Lemma 3.6.** *If $B$ is an absorbing set for $P$, then $B$ is absorbing for $P^n$ for all $n \geq 1$.*

Proof. The proof is by induction. By hypothesis, $P(1_B)(z) = 1$ for all $z \in B$. Therefore $P(1_B) - 1_B \geq 0$. Since $P$ is positive,

$$1 \geq P^2(1_B)(z) = P(P(1_B))(z) \geq P(1_B)(z) = 1$$

for all $z \in B$. Thus $B$ is absorbing for $P^2$. The induction is now clear. □

**Definition 11.** We say that a Borel set $B$ is *strongly absorbing* for the Markov operator $P$ if the following condition holds: A point $x$ belongs to the complement of $B$ if and only if there is an open neighborhood $V$ of $x$ and a an open neighborhood $W$ of $B$ so that $\int_V p(dx, y) = 0$ for all $y \in W$.

Note that a closed strongly absorbing set for $P$ is absorbing. Indeed, if we fix $y \in B$, we can cover the complement of $B$ with a countable collection of open sets $\{V_n\}_{n \geq 0}$ so that $\int_{V_n} p(dx, y) = 0$ for all $n \in \mathbb{N}$. Then $\int_{B^c} p(dx, y) = 0$ so $\int_B p(dx, y) = 1$. Thus $B$ is absorbing.

We come now to the main result of this section.

**Theorem 3.7.** *Suppose $P$ is a Markov operator on $C(X)$ such that $E(P)$ satisfies condition (L). Then $\mathcal{O}(P)$ is simple if and only if the only closed strongly absorbing set is $X$.*

PROOF. According to Theorem 10.2 of [31] to show that $\mathcal{O}(P)$ is simple, we must show that the only open saturated hereditary sets in $X$ are $X$ and $\emptyset$. Consequently, it is enough to show that a nonempty open subset $U$ of $X$ is a saturated hereditary subset of $E^0$ if and only if $K := X \setminus U$ is a closed strongly absorbing set for $P$.

Assume first that $U$ is an open saturated hereditary set. Then since $K$ is closed, $K$ is compact. Further, since $U$ is hereditary, $r(s^{-1}(U)) \subset U$. So, if $x \in U$, then for each $y \in K$ there is an open neighborhood $V_y$ of $x$ and an open neighborhood $W_y$ of $y$ so that $\int_{V_y} p(dx, z) = 0$ for all $z \in W_y$. Consequently, $\{W_y\}_{y \in K}$ is an open cover of $K$. Let $\{W_{y_1}, W_{y_2}, \dots, W_{y_n}\}$ be a finite subcover of $K$ and let $\{V_{y_1}, V_{y_2}, \dots, V_{y_n}\}$ be the corresponding open neighborhoods of $x$. Then for $V = \bigcap_{i=1}^n V_{y_i}$ and $W = \bigcup_{i=1}^n W_{y_i}$ ,

$$\int_V p(dx, y) = 0 \text{ for all } y \in W.$$

Thus $K$ is strongly absorbing.

Assume, conversely, that $K$ is a strongly absorbing subset of $X$. If $U$ is not hereditary, there is $x \in U$ and $y \in K$ so that $(x, y) \in \operatorname{supp} P$. Therefore for any neighborhood $V_x$ of $x$ and any neighborhood $W_y$ of $y$ there is $z \in W_y$ so that $\int_{V_x} p(dx, y) > 0$. This contradicts the definition of a strongly absorbing set. Thus $U$ is hereditary. To see that $U$ is saturated, let $x$ be a regular vertex with the property that $r(s^{-1}(x)) \subset U$. Then for any $y \in K$ there is a neighborhood $V_y$ of $x$ and a neighborhood $W_y$ of $y$ so that $\int_{V_y} p(dx, z) = 0$ for all $z \in V_y$. Since $E^0_{reg} = E^0_{fin} - E_{sinks}$, we may assume without loss of generality that each of the $V_y$ is contained in $E^0_{reg}$. By the compactness of $K$ again, we can find a finite number of points such that $\{W_{y_1}, W_{y_2}, \dots, W_{y_n}\}$ is a cover of $K$. If $V = \bigcap_{i=1}^n V_{y_i}$ as before, we obtain an open neighborhood $V$ of $x$ and an open neighborhood $W = \bigcup_{i=1}^n W_{y_i}$ of $K$ so that $\int_V p(dx, y) = 0$ for all $y \in W$. Since $K$ is strongly absorbing it follows that $x \in U$, thus $U$ is an open saturated hereditary set and the theorem is proved. □

## 4. EXAMPLES

4.1. **Finite Markov Chains.** Let $X$ be a finite set and $P$ be a Markov operator on $C(X)$. Recall from Example 2.5 that $P$ is given by (the transpose of) a stochastic matrix $\{p_{ij}\}$. Then $\mathcal{O}(P)$ is simple if and only if the Markov chain is ergodic ([28]). We recover, thus, Theorem 5.16 of [34].

4.2. **Homeomorphisms and local homeomorphism.** Let $X$ be a compact second countable Hausdorff space and let $\tau$ be a homeomorphism or a local homeomorphism on $X$. The canonical Markov operators we defined in Examples 2.6 and 2.7 are defined by

$$P(f)(y) = f(\tau^{-1}(y))$$

if $\tau$ is a homeomorphism, and

$$P(f)(y) = \frac{1}{|\tau^{-1}(y)|} \sum_{x \in \tau^{-1}(y)} f(x)$$

if $\tau$ is a local homeomorphism. In both cases the support of $P$ is the graph of $\tau$. Let $E(P) = (E^0, E^1, r, s, \lambda)$ be the associated topological quiver. We describe next a characterization of condition (L) for this class of examples. We begin with a straightforward application of Proposition 3.1.

**Corollary 4.1.** *If $P$ is the Markov operator associated with a homeomorphism or a local homeomorphism $\tau$ on $X$, and $x, y \in X$, then there is a path from $x$ to $y$ if and only if $y = \tau^n(x)$ for some $n \geq 1$. Thus $x$ is a base point of a loop if and only if $x$ is a fixed point for $\tau^n$ for some $n \geq 1$.*

Since $(x, y) \in E^1$ if and only if $\tau(x) = y$, it follows that for any point $x \in E^0$ there is exactly one edge whose source is $x$. In particular no loop has an exit. Thus we obtain the following characterization for condition (L).

**Corollary 4.2.** *Let $\tau$ be a homeomorphism or a local homeomorphism on $X$. Let $P$ be the associated Markov operator. Then the topological quiver $E(P)$ satisfies condition (L) if and only if the set of fixed points of $\tau^n$, $n \geq 1$, has empty interior.*

One can easily see that a subset $U$ of $X$ is hereditary for $E$ if and only if $\tau^{-1}(K) \supset K$, where $K = X \setminus U$. Moreover $U$ is a saturated hereditary subset of $X$ if and only if $K = \tau^{-1}(K)$. Thus a set $K$ is strongly absorbing if and only if $K$ is invariant under $\tau$, that is $K = \tau^{-1}(K)$. Then we obtain the following characterization for the simplicity of these $C^*$-algebras, which in the case of homeomorphisms, at least, has been known for quite some time.

**Proposition 4.3.** *If $P$ is the Markov operator associated to a homeomorphism or local homeomorphism $\tau$ on $X$, then $\mathcal{O}(P)$ is simple if and only if the set of fixed points of all positive powers of $\tau$ has empty interior and there are no closed invariant sets under $\tau$.*

4.3. **Iterated Function Systems.** Let $X$ be a compact metric space. Recall from Example 2.8 that an iterated function system (i.f.s.) is a collection of injective contractions $\{f_1, f_2, \ldots, f_N\}$ with $N \geq 2$. We assume that $X$ is invariant for the i.f.s., that is

$$X = f_1(X) \bigcup f_2(X) \bigcup \cdots \bigcup f_N(X).$$

The canonical Markov operator associated with an i.f.s. is

$$P(f)(y) = \frac{1}{N} \sum_{i=1}^{N} f \circ f_i(y)$$

and its support is $\bigcup_{i=1}^N \operatorname{cograph} f_i$. The associated topological quiver is given by $E^0 = X$, $E^1 = \bigcup_{i=1}^N \operatorname{cograph} f_i$, $s(x,y) = x$, $r(x,y) = y$. We will show that these topological quivers will always satisfy condition (L) (Proposition 4.7). We start with an immediate consequence of Proposition 3.1.

**Corollary 4.4.** *If $P$ is the Markov operator associated with an iterated function system $(f_1, f_2, \ldots, f_N)$ on $X$, and $x, y \in X$ then there is a path from $x$ to $y$ if and only if there is a finite word $w = w_1 w_2 \ldots w_n$, $w_i \in \{1, \ldots, N\}$, such that $y = f_w(x)$, where $f_\omega = f_{\omega_1} \circ \cdots \circ f_{\omega_n}$. Thus $x$ is a base point for a loop if and only if $x$ is the fixed point of $f_\omega$, for some finite word $\omega \in \{1, \ldots, N\}^n$.*

Proof. For $f \in C(X)$ we have that

$$P^n(f)(y) = \frac{1}{N^n} \sum_{w \in \{1,\ldots,N\}^n} f \circ f_w(y).$$

Now Proposition 3.1 implies the conclusion. □

Since $X$ is the invariant set of the iterated function system it follows that if $x \in X$ then there is some $y \in X$ and $i \in \{1, \ldots, N\}$ such that $x = f_i(y)$. Therefore $(x, y)$ belongs to the support of $P$ and there is at least one edge with source $x$. We identify next the classes of iterated function systems for which there is exactly one edge with source $x$ for all $x \in X$. These i.f.s. are classified as follows.

The iterated function system is called *totally disconnected* if $f_i(X) \bigcap f_j(X) = \emptyset$ for $i \neq j$. In this case, there is a local homeomorphism $\tau : X \to X$ such that $\tau \circ f_i = 1_X$ and the Markov operator $P$ is the same as the Markov operator associated to the local homeomorphism $\tau$.

A point $x \in X$ is called a *branch point (see* [24, Definition 2.4], [23, Definition 2.4]*)* if there are two indices $i \neq j$ and $y \in X$ such that $x = f_i(y) = f_j(y)$. If $f_i(X) \bigcap f_j(X)$ is not empty but consists of only branch points, then there still is a continuous map $\tau$ such that $\tau \circ f_i = 1_K$. The map $\tau$ is a branch covering in this case.

**Proposition 4.5.** *Let $(f_1, f_2, \ldots, f_n)$ be an iterated function system with invariant set $X$ and let $P$ the associated Markov operator on $C(X)$. Then for any point $x \in E^0$ there is exactly one edge whose source is $x$ if and only if the iterated function system is totally disconnected or if $f_i(X) \bigcap f_j(X) \neq \emptyset$ then it contains only branch points.*

Proof. If the iterated function system is totally disconnected, given any $x \in X$ there is a unique $y \in X$ and $i \in \{1, \ldots, N\}$ such that $x = f_i(y)$. That is, there is a unique $y \in X$ with $(x, y) \in \operatorname{supp} P$. If the iterated function system is not totally disconnected and $f_i(X) \bigcap f_j(X)$ is either empty or contains branch points, then for any $x \in X$ there is a *unique* $y \in X$ such that $x = f_i(y)$ for some index $i$ which might not be unique. Then $(x, y)$ is the unique edge in $E^1$ with source $x$.

Conversely, if for any $x \in X$ there is a unique edge in $E^1$ with source $x$, then there is a unique $y \in X$ such that $(x, y)$ belongs to the support of $P$. Therefore for each $x$ there is a unique $y$ so that $x = f_i(y)$ for some $i \in \{1, \ldots, N\}$. This clearly implies that the iterated function system is either totally disconnected or $f_i(X) \bigcap f_j(X)$ consists only of branch points, if it is nonempty. □

**Example 4.6.** Let $X = [0, 1]$, $f_1(x) = \frac{1}{2}x$, $f_2(x) = 1 - \frac{1}{2}x$. Then the union of the cographs of $f_i$ is the graph of the tent map ([24, Example 4.5]) and $f_1(X) \bigcap f_2(X)$ contains only $1/2$, which is a branch point. Thus for each point $x \in [0, 1]$ there is a unique edge with source $x$. Note, however, that $1/2$ is *not* a finite emitter in the sense of [31, Definition 3.14]. According to [24, Proposition 2.6], [23, Proposition 2.6], the set of finite emitters for this example is $[0, 1] \setminus \{1/2\}$.

It follows that if the iterated function system is either totally disconnected or $f_i(X) \bigcap f_j(X)$ contain only branch points ($i \neq j$), no loop has an exit. Nevertheless, the associated topological quiver will always satisfy condition (L).

**Proposition 4.7.** *Assume that $X$ is a non-discrete uncountable compact metric space and $(f_1, f_2, \ldots, f_N)$ is an iterated function system on $X$. Let*

$$Pf(y) = \frac{1}{N} \sum_i f \circ f_i(y) \tag{5}$$

*be the associated Markov operator. Then the corresponding topological quiver $E(P)$ satisfies condition (L).*

Proof. We will prove that the set of base point of loops has empty interior. Recall from Lemma 4.4 that $x$ is the base point of a loop in $X$ if and only if $x$

is a fixed point of $f_w$ for a finite word $w \in \{1, \ldots, N\}^n$, for some $n \geq 1$. Since each $f_w$ is a contraction it has a unique fixed point. Since the number of finite words over $\{1, \ldots, N\}$ is countable so is the set of base points of loops, hence it has empty interior. □

Next we study the open saturated hereditary subsets of $E^0$ for an iterated function system. The lack of the existence of such sets is an example of the "rigidity" of iterated function systems in operator algebras. For other results suggesting this rigidity see [33] and [22].

**Proposition 4.8.** *Let $(f_1, f_2, \ldots, f_N)$ be an iterated function system with invariant set $X$. Then the only two open hereditary subsets are $X$ and the empty set.*

PROOF. Suppose that $U$ is an open hereditary set which is not $X$ or the empty set. Then $K = X \setminus U$ is a nonempty compact subset of $X$. We claim that

$$\bigcup_{i=1}^{N} f_i(K) \subseteq K. \tag{6}$$

To prove this claim let $x \in K$ and assume that $f_i(x) \in U$ for some $i$. Then $x \in r(s^{-1}(f_i(x)) \subset U$, since $(f_i(x), x)$ belongs to the support of $P$. This is a contradiction and the above inclusion holds.

We let $F$ be the map defined for all non-empty compact subsets of $K$ via the formula $F(A) = f_1(A) \bigcup \cdots \bigcup f_N(A)$. Then we can rewrite equation (6) as $F(K) \subset K$. It follows that $F^n(K) \subseteq K$ for all $n \geq 1$. The sequence $\{F^n(K)\}_n$ converges to $X$ in the Hausdorff metric (see [21, 2, 14]). Thus $X = K$ and this is a contradiction. □

It follows from Theorem 3.7 together with Propositions 4.7 and 4.8 that the $C^*$-algebra $\mathcal{O}(P)$ associated to an iterated function system is always simple. We record this fact in the following proposition.

**Proposition 4.9.** *Suppose that $X$ is a compact nondiscrete metric space and that $(f_1, f_2, \ldots, f_N)$ is an iterated function system on $X$. Let $P$ be the Markov operator associated to this i.f.s. via* (5)*. Then $\mathcal{O}(P)$ is simple.*

4.4. **Collection of continuous maps.** Let $X = [0, 1]$, $f_1(x) = x$, and $f_2(x) = 1 - x$. We associated in Example 2.9 the following Markov operator

$$P(f)(y) = \frac{1}{2} \sum_{i=1}^{2} f \circ f_i(y).$$

Then the support of $P$ is the union of the cographs of $f_i$, $i = 1, 2$. The formulas defining the topological quiver and the associated $C^*$-correspondence are identical with those for iterated function systems. The similarities end here, however. Every point $x \in X$ is a base point for at least one loop. All nonreturning loops (see [31, Definition 6.5]) have length at most two. For $x \in X$, $x \neq 1/2$, $(x, x)$ and $((x, 1-x), (1-x), x))$ are the only nonreturning loops. If $x \neq 1/2$ the $s^{-1}(x)$ contains exactly two edges, $(x, x)$ and $(x, 1-x)$. If $\alpha$ is a finite path so that $r(\alpha) = 1/2$ then $s(\alpha) = 1/2$. The topological quiver satisfies condition (L) since the set of base points of loops with no exits is $\{1/2\}$ and thus it has empty interior in $[0, 1]$. The $C^*$-algebra $\mathcal{O}(P)$ is not simple, however, because there are many strongly absorbing closed subsets of $[0, 1]$. For example, the sets $\{x, 1-x\}$, $x \neq 1/2$, and $\{1/2\}$ are all closed strongly absorbing sets for $P$.

4.5. **Independent Random Variables.** Recall the Markov operator we defined in Example 2.10: $X = [0, 1]$, $p(\cdot, y) = m$, Lebesgue measure, for all $x \in [0, 1]$. Then

$$P(f)(y) = \int_{[0,1]} f(x) dm(x),$$

and the support of $P$ is $[0, 1] \times [0, 1]$. Since $s^{-1}(x) = \{x\} \times [0, 1]$ for all $x \in [0, 1]$, every point is a base point of a loop and every loop has an exit. Therefore the topological quiver satisfies condition (L). Since the only absorbing closed set is $[0, 1]$ it follows from Theorem 3.7 that $\mathcal{O}(P)$ is simple.

## 5. From Quivers to Markov Operators

We have seen above that a Markov operator determines a topological quiver. In this section we want to prove that many $C^*$-algebras associated with topological quivers are isomorphic with $C^*$-algebras of specific Markov operators. The key ingredient in our proof is the so-called *dual topological quiver*.

Let $E = (E^0, E^1, r, s, \lambda)$ be a topological quiver, with $E^0$ and $E^1$ locally compact spaces. We define it's dual as follows. Set $\widehat{E}^0 = E^1$, $\widehat{E}^1 = E^1 * E^1$ - the set of paths of length 2 in $E$, $\widehat{s}(e_1, e_2) = e_1$, $\widehat{r}(e_1, e_2) = e_2$. We claim that $\widehat{r}$ is an open map. Let $U_1$ and $U_2$ be open in $E^1$. Since $\widehat{r}(U_1 * U_2) = s^{-1}(r(U_1)) \bigcap U_2$ and $r$ is an open map it follows that $\widehat{r}(U_1 * U_2)$ is open in $\widehat{E}^0$. Thus $\widehat{r}$ is open. The family of Radon measures is defined by $\widehat{\lambda}_{e_2} = \lambda_{s(e_2)} \times \delta_{e_2}$, that is, $\int f(u_1, u_2) d\widehat{\lambda}_{e_2}(u_1, u_2) = \int f(u_1, e_2) d\lambda_{s(e_2)}(u_1)$.

If $E^0$ and $E^1$ are compact spaces, $r$ is surjective, and $\lambda_v$ are probability measures for all $v \in E^0$, then we define a Markov operator $P : C(\widehat{E}^0) \to C(\widehat{E}^0)$

via

$$Pf(e_2) = \int f(e) d\lambda_{s(e_2)}(e).$$

The topological quiver defined by $P$ is $(\widehat{E}^0, \widehat{E}^1, \widehat{r}, \widehat{s}, \widehat{\lambda})$. We will show that $C^*(E)$ and $C^*(\hat{E})$ are isomorphic $C^*$-algebra under the assumption that $E$ has no sinks and no infinite emitters.

**Theorem 5.1.** *Let $E = (E^0, E^1, r, s, \lambda)$ be a topological quiver with no sinks and no infinite emitters, that is assume $E^0_{reg} = E^0$. If $\widehat{E} = (\widehat{E}^0, \widehat{E}^1, \widehat{r}, \widehat{s}, \lambda)$ is the dual quiver of $E$, then $C^*(E)$ and $C^*(\widehat{E})$ are isomorphic.*

Proof. Let $(i, \psi)$ and $(\widehat{i}, \widehat{\psi})$ be the universal representations of $C^*(E)$ and $C^*(\widehat{E})$, respectively. Let $A = C_0(E^0)$, $\mathcal{X} = \overline{C_c(E^1)}$ be the $C^*$-correspondence associated with $E$, and let $\widehat{A} = C_0(\widehat{E}^0) = C_0(E^1)$ and $\widehat{\mathcal{X}} = \overline{C_c(\widehat{E}^1)}$ be the $C^*$-correspondence associated with $\widehat{E}$.

An element $f \in C_b(E^1)$ determines an adjointable operator $T_f$ on $\mathcal{X}$ by $T_f(\xi)(e) = f(e)\xi(e)$ (see also [31, Lemma 3.6]). Note that $T_f^* = T_{f^*}$. We define a representation $(\pi, V)$ of $(\widehat{A}, \widehat{\mathcal{X}})$ into $C^*(E, \lambda)$ by the formulae: $\pi(f) = i^{(1)}(T_f)$ for $f \in \widehat{A} = C_0(E^1)$ and $V(f_1 * f_2) = \psi(f_1) i^{(1)}(T_{f_2})$ for $f_1, f_2 \in C_c(E^1)$, where $f_1 * f_2(e_1, e_2) = f_1(e_1) f_2(e_2)$. (Recall that $i^{(1)}$ is the extension of $i$ to $\mathcal{L}(\mathcal{X})$ defined using equation (3) and the subsequent discussion.) We claim that $(\pi, V)$ is a Cuntz-Pimsner representation. Let $f \in \widehat{A}$ and $f_1, f_2 \in C_c(E^1)$. Then using equation (4) (with $\pi$ replaced by $i$),

$$\begin{aligned} V(f \cdot f_1 * f_2) &= V(T_f f_1 * f_2) = \psi(T_f f_1) i^{(1)}(T_{f_2}) \\ &= i^{(1)}(T_f)\psi(f_1)\widehat{i}^{(1)}(T_{f_2}) = \pi(f) V(f_1 * f_2), \\ V(f_1 * f_2 \cdot f) &= \psi(f_1)\widehat{i}^{(1)}(T_{f_2 f}) = \psi(f_1) i^{(1)}(T_{f_2} T_f) \\ &= \psi(f_1) i^{(1)}(T_{f_2})\widehat{i}^{(1)}(T_f) = V(f_1 * f_2)\pi(f). \end{aligned}$$

If $f_1, f_2, g_1, g_2 \in C_c(E^1)$ note that $\langle f_1 * f_2, g_1 * g_2 \rangle_{\widehat{A}}(e) = \overline{f_2(e)} \langle f_1, g_1 \rangle_A (s(e)) g_2(e)$ and $\langle f_1, g_1 \rangle_A \circ s \in C_b(E^1)$. Thus

$$\pi\big(\langle f_1 * f_2, g_1 * g_2 \rangle_{\widehat{A}}\big) = \pi(f_2^*) i^{(1)}(T_{\langle f_1, g_1 \rangle_A \circ s}) \pi(g_2).$$

Moreover

$$i^{(1)}(T_{\langle f_1, g_1 \rangle_A \circ s}) \psi(\xi) = \psi(\langle f_1, g_1 \rangle_A \cdot \xi) = i(\langle f_1, g_1 \rangle_A)\psi(\xi) = \psi(f_1)^* \psi(g_1) \psi(\xi).$$

Thus $i^{(1)}(T_{\langle f_1, g_1 \rangle \circ s}) = \psi(f_1)^* \psi(g_1)$. This implies that

$$V(f_1 * f_2)^* V(g_1 * g_2) = \pi\big(\langle f_1 * f_2, g_1 * g_2 \rangle_{\widehat{A}}\big).$$

Thus $(\pi, V)$ is a Toeplitz representation.

We show next that $(\pi, V)$ is a Cuntz-Pimsner covariant representation. Since $E^0 = E^0_{reg}$ it follows from [31, Proposition 3.15] and [31, Theorem 3.11] that if $f \in C_0(\widehat{E}^0_{reg})$ then $T_f \in \mathcal{K}(\mathcal{X})$. Moreover, if $f \in C_c(\widehat{E}^0_{reg})$ we can use the following construction from the proof of Theorem 3.11 of [31]. If $K_f$ is the support of $f$ there is a finite cover $\{ U_i \}_{i=1}^n$ such that $K_f \subseteq \bigcup_{i=1}^n U_i$ and $r|_{U_i} : U_i \to s(U_i)$ is a homeomorphism. Moreover we can choose $U_i$ such that $\overline{U}_i$ is compact, $s^{-1}(r(\overline{U}_i))$ is compact and $r$ restricted to $V_i := s^{-1}(r(U_i))$ is a homeomorphism onto its image (see [31, Proposition 3.15]). If $\{ \zeta_i \}_{i=1}^n$ is a partition of unity on $K_f$ subordinate to $\{ U_i \}_{i=1}^n$ and $\xi_i = f\zeta_i^{1/2}$ and $\eta_i(\alpha) := \zeta_i^{1/2}(\alpha)(\lambda_{r(\alpha)}(\{\alpha\}))^{-1}$ for $\alpha \in E^1$ then $\xi_i, \eta_i \in C_c(E^1)$, $\operatorname{supp}\xi_i = \operatorname{supp}\eta_i \subseteq U_i$, and $T_f = \sum_{i=1}^n \Theta_{\xi_i,\eta_i}$. Moreover, since $s^{-1}(r(K_f))$ is a closed subset of $\bigcup_{i=1}^n s^{-1}(r(\overline{U}_i))$ which is compact, it follows that $s^{-1}(r(K_f))$ is compact and $\{ V_i \}_{i=1}^n$ is an open cover of $s^{-1}(r(K_f))$. Let $\{ \varsigma_i \}_{i=1}^n$ be a partition of unity subordinate to $\{ V_i \}_{i=1}^n$. Then, if $h_j = \varsigma_j^{1/2}$, we find that for $f_1, f_2 \in C_c(E^1)$ and $(e_1, e_2) \in \widehat{E}^1$,

$$
\begin{aligned}
\widehat{\Phi}(f)(f_1 * f_2)(e_1, e_2) &= f(e_1)f_1(e_1)f_2(e_2) = \sum_{i,j} f(e_1)\zeta_i(e_1)f_1(e_1)\varsigma_j(e_2)f_2(e_2) \\
&= \sum_{i,j} \xi_i(e_1)\eta_i(e_1)\lambda_{r(e_1)}(\{e_1\})f_1(e_1)h_j(e_2)h_j(e_2)f_2(e_2) \\
&= \sum_{i,j} \xi_i(e_1)h_j(e_2)\int_{r^{-1}(r(e_1))} \eta_i(u)h_j(e_2)f_1(u)f_2(e_2)d\lambda_{s(e_2)}(u) \\
&= \sum_{i,j} \xi_i * h_j(e_1, e_2)\langle \eta_i * h_j, f_1 * f_2\rangle_{\widehat{A}}(e_2) \\
&= \sum_{i,j} \big(\xi_i * h_j \cdot \langle \eta_i * h_j, f_1 * f_2\rangle_{\widehat{A}}\big)(e_1, e_2) \\
&= \sum_{i,j} \Theta_{\xi_i * h_j, \eta_i * h_j}(f_1 * f_2)(e_1, e_2).
\end{aligned}
$$

In this computation we used the fact that $r$ is a homeomorphism on $\operatorname{supp}\xi_i = \operatorname{supp}\eta_i \subset U_i$ and that $r(e_1) = s(e_2)$.Thus

$$
\widehat{\Phi}(f) = \sum_{i=1}^n \sum_{j=1}^n \Theta_{\xi_i * h_j, \eta_i * h_j}.
$$

Therefore

$$\begin{aligned}\pi(f) = i^{(1)}(T_f) &= \sum_{i=1}^{n} i^{(1)}(\Theta_{\xi_i,\eta_i}) = \sum_{i=1}^{n} \psi(\xi_i)\psi(\eta_i)^* \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} \psi(\xi_i) i^{(1)}(T_{\varsigma_j})\psi(\eta_i)^* = \sum_{i,j=1}^{n} \psi(\xi_i) i^{(1)}(T_{h_j}) i^{(1)}(T_{h_j})^* \psi(\eta_i)^* \\ &= \sum_{i,j=1}^{n} V(\xi_i * h_j) V(\eta_i * h_j)^* = \pi^{(1)}\big(\widehat{\phi}(f)\big).\end{aligned}$$

Using also [31, Lemma 3.10] $(\pi, V)$ is a Cuntz-Pimsner covariant representation. Therefore there exists a $*$-homomorphism $\pi \rtimes V : C^*(\widehat{E}) \to C^*(E)$ such that $\pi \rtimes V \circ \widehat{i} = \pi$ and $\pi \rtimes V \circ \widehat{\psi} = V$.

We prove that since $E^0_{reg} = E^0$, $\pi \rtimes V$ is surjective. If $a \in C_0(E^0)$ then $a \circ s \in C_0(E^1)$ ([31, Corollary 3.12]) and $\phi(a) = T_{a\circ s}$ . Therefore $i(a) = \pi(a \circ s)$ and $i(a) \in \pi(\widehat{A})$. Suppose now that $\xi \in C_c(E^1)$. Let $\{U_i\}_{i=1}^n$ be an open cover of the support of $\xi$ and $\{\zeta_i\}_{i=1}^n$ be a partition of unity subordinate to $\{U_i\}$. Then

$$\begin{aligned}\psi(\xi) &= \sum_i \psi(\xi) i(\zeta_i) = \sum_i \psi(\xi) i^{(1)}(\phi(\zeta_i)) \\ &= \sum_i \psi(\xi) i^{(1)}(T_{\zeta_i \circ s}) = \sum_i V(\xi * \zeta_i \circ s).\end{aligned}$$

Thus $\psi(\xi) \in V(\widehat{\mathcal{X}})$ and it follows that $\pi \rtimes V$ is surjective.

Finally, we use the gauge invariance uniqueness theorem to show the injectivity of $\pi \rtimes V$. Let $\gamma_z$ and $\beta_z$ be the gauge action on $C^*(E)$ and $C^*(\widehat{E})$, respectively. Recall that $\gamma_z(i(a)) = i(a)$ for all $a \in C_0(E^0)$ and $\gamma_z(\psi(\xi)) = z\psi(\xi)$ for all $\xi \in \mathcal{X}$; similar statements hold for $\beta_z$. Let $f \in C_0(E^1)$. Then

$$\gamma_z \circ \pi \rtimes V(\widehat{i}(f)) = \gamma_z(\pi(a)) = \gamma_z(i^{(1)}(T_f))$$

which, since $\gamma_z(i^{(1)}(T_f))(\psi(\xi)) = \frac{1}{z}\gamma_z\big(i^{(1)}(T_f)\psi(\xi)\big) = \frac{1}{z}\gamma_z(\psi(T_f\xi)) = \psi(T_f\xi) = i^{(1)}(T_f)\psi(\xi)$, equals

$$i^{(1)}(T_f) = \pi(f) = \pi \rtimes V(\widehat{i}(f)) = \pi \rtimes V(\beta_z(\widehat{i}(a)).$$

If $f_1, f_2 \in C_c(E^1)$

$$\begin{aligned}\gamma_z \circ \pi \rtimes V(\widehat{\psi}(f_1 * f_2)) &= \gamma_z(V(f_1 * f_2)) = \gamma_z\big(\psi(f_1) i^{(1)}(T_{f_2})\big) \\ &= z\psi(f_1) i^{(1)}(T_{f_2}) = \pi \rtimes V(z\widehat{\psi}(f_1 * f_2)) \\ &= \pi \rtimes V \circ \beta_z(\widehat{\psi}(f_1 * f_2).\end{aligned}$$

Thus $\pi \rtimes V$ is a $*$-isomorphism. $\square$

**Corollary 5.2.** *Suppose $E = (E^0, E^1, r, s, \lambda)$ is a topological quiver with $E^0$ and $E^1$ compact spaces, $r$ surjective, and $\lambda_v$ a probability measure for all $v \in E^0$. If there are no infinite emitters, then there is a Markov operator $P$ such that $C^*(E)$ is $*$-isomorphic with $\mathcal{O}(P)$.*

*Remark* 5.3. Brenken proved in [6, Theorem 4.8] that if $E$ is a proper, range finite topological quiver such that $F_G = r(E^1) - \overline{s(E^1)}$ closed in $r(E^1)$, then there is a topological relation $\widehat{E} = (\widehat{E}^0, \widehat{E}^1, \widehat{r}, \widehat{s}, \widehat{\lambda})$ such that $C^*(E)$ is isomorphic to $C^*(\widehat{E})$. His topological relation coincides with our dual topological quiver in the absence of sinks. Our proof is different, though, and our results imply that the dual topological quiver comes from a Markov operator, under suitable hypotheses.

*Remark* 5.4. Brenken also showed in [6] that without the requirement that there are no infinite emitters, then Theorem 5.1 can fail. If $E^0 = \{v\}$, $E^1 = [0, 1]$, $r(x) = s(x) = v$ for all $x \in E^1$, $\lambda_v = m$, Lebesgue measure, then it's dual topological quiver $\widehat{E}$ is given by the Markov operator described in Example 2.10. Brenken describes these $C^*$-algebras in the comments following Corollary 4.9 of [6]. He proves that $\Phi(C(E^0)) \bigcap \mathcal{K}(\mathcal{X}) = \emptyset$, $C^*(E)$ is isomorphic to $\mathcal{O}_\infty$, with $K_0$ group $\mathbb{Z}$ and trivial $K_1$ group, and $C^*(\widehat{E})$ is a unital Kirchberg algebra with both $K$ groups $\mathbb{Z}$. Thus $C^*(E)$ and $C^*(\widehat{E})$ are not even Morita equivalent, let alone isomorphic. Nevertheless, one can speculate if Corollary 5.2 is true without the hypothesis that there are no infinite emitters. The point is that in this case, the Markov operator will not necessarily come from the dual quiver.

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(Ionescu) Department of Mathematics, 196 Auditorium Road, Unit 3009 University of Connecticut, Storrs, CT 06269-3009

*E-mail address*: `ionescu@math.uconn.edu`

(Muhly) Department of Mathematics, The University of Iowa, Iowa City, IA 52242-1419

*E-mail address*: `pmuhly@math.uiowa.edu`

(Vega) Department of Mathematics, St. Ambrose University, 518 W. Locust St, Davenport, IA 52803

*E-mail address*: `VegaVictorM@sau.edu`