The Generalized Effros-Hahn Conjecture for Groupoids

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ABSTRACT. The generalized Effros-Hahn conjecture for groupoid C^*-algebras says that, if G is amenable, then every primitive ideal of the groupoid C^*-algebra $C^*(G)$ is induced from a stability group. We prove that the conjecture is valid for all second countable amenable locally compact Hausdorff groupoids. Our results are a sharpening of previous work of Jean Renault and depend significantly on his results.

1. INTRODUCTION

A dynamical system (A, G, α), where A is a C^*-algebra, G is a locally compact group and α is a strongly continuous homomorphism of G into $\operatorname{Aut} A$, is called *EH-regular* if every primitive ideal of the crossed product $A \rtimes_\alpha G$ is induced from a stability group (see [19, Definition 8.18]). In their 1967 *Memoir* [4], Effros and Hahn conjectured that if (G, X) was a second countable locally compact transformation group with G amenable, then $(C_0(X), G, \mathrm{lt})$ should be EH-regular. This conjecture, and its generalization to dynamical systems, was proved by Gootman and Rosenberg in [6] building on results due to Sauvageot [17, 18]. For additional comments on this result, its applications, as well as a precise statement and proof, see [19, Section 8.2 and Chapter 9].

In [16], Renault gives the following version of the Gootman-Rosenberg-Sauvageot Theorem for groupoid dynamical systems. Let G be a locally compact groupoid and (A, G, α) a groupoid dynamical system. If R is a representation of the crossed product $A \rtimes_\alpha G$, then Renault forms the restriction, $\hat{L}$, of R to the isotropy groups of G and forms an induced representation $\operatorname{Ind} \hat{L}$ of $A \rtimes_\alpha G$ such that $\ker R \subset \ker(\operatorname{Ind} \hat{L})$ [16, Theorem 3.3]. When G is suitably amenable, then the reverse conclusion holds [16, Theorem 3.6]. This is a powerful result and allows Renault to establish some very striking results concerning the ideal structure of crossed products and has deep applications to the question of when a crossed product is simple (see [16, Section 4]).

In this note, our object is to provide a significant sharpening of Renault's result in the case of a groupoid C^*-algebra—that is, a dynamical system where G acts on the commutative algebra $C_0(G^{(0)})$ by translation. We aim to show that if G is Hausdorff and amenable, then every primitive ideal K of $C^*(G)$ is induced from a stability group. That is, we show that $K = \operatorname{Ind}_{G(u)}^G J$ for a primitive ideal J of $C^*(G(u))$, where $G(u)$ is the stability group at some $u \in G^{(0)}$. This not only provides a cleaner generalization of the Gootman-Rosenberg-Sauvageot result to the groupoid setting, but gives us a much better means to study the fine ideal structure of groupoids and the primitive ideal space (together with its topology) in particular. (For further discussion of these ideas, see [2, Section 4].)

In Section 2 we give a careful statement of the main result, and give a brief summary of some of the tools and ancillary results needed in the sequel. Since the proof of the main result is rather involved, we also give an overview of the proof to make the subsequent details easier to parse. Then in Section 3, we give the proof itself. Our techniques require that we work whenever possible with separable C^*-algebras. Therefore all our groupoids are assumed to be second countable. We also assume that our locally compact groupoids have continuous Haar systems and are Hausdorff. We also assume that homomorphisms between C^*-algebras are $*$-preserving and that representations are nondegenerate.

2. THE MAIN RESULT

Unlike the situation for groups, the definition of amenability of a locally compact groupoid is a bit controversial. The currently accepted definition originally comes from [14, p. 92]: a locally compact groupoid G with continuous Haar system $\lambda = \{\lambda^u\}$ is *amenable* if there is a net $\{\varphi_i\} \subset C_c(G)$ such that

(1) the functions $u \mapsto \int_G |\varphi_i(\gamma)|^2 \, d\lambda^u(\gamma)$ are uniformly bounded, and
(2) the functions $\gamma \mapsto \varphi_i * \varphi_i^*(\gamma)$ converge to the constant function 1 uniformly on compacta.

If G is a group, then we recover the usual notion of amenability (for example, see [19, Proposition A.17]). A different definition of amenability for a locally compact groupoid is given in [1, Definition 2.2.8].[1] Fortunately, [1, Proposition 2.2.13(iv)] implies the two definitions are equivalent (and gives some additional equivalent conditions). In particular, [1, Theorem 2.2.13] implies that amenability is preserved under equivalence of groupoids as defined in [9, Definition 2.1]. Thus the notion of amenability is independent of the choice of continuous Haar system on G.

[1] Both the numbering and the statements of some results in the published version of this paper differ from those in the widely circulated preprint.

Theorem 2.1. *Assume that G is a second countable locally compact Hausdorff groupoid with Haar system $\{\lambda^u\}_{u\in G^{(0)}}$. Assume also that G is amenable. If $K \subset C^*(G)$ is a primitive ideal, then K is induced from an isotropy group. That is,*

$$K = \operatorname{Ind}_{G(u)}^{G} J$$

for a primitive ideal $J \in \operatorname{Prim}(C^*(G(u))$.

Remark 2.2. Let R be the equivalence relation on $G^{(0)}$ induced by G: $r(\gamma) \sim s(\gamma)$ for all $\gamma \in G$. We give R the Borel structure coming from the topology on R induced from G (which is often finer than the product topology on R viewed as a subset of $G^{(0)} \times G^{(0)}$). Since the proof of Theorem 2.1 requires only that we are entitled to apply Renault's [16, Theorem 3.6], Theorem 2.1 is valid under the weaker assumption that the Borel equivalence relation R is amenable with respect to every quasi-invariant measure μ on $G^{(0)}$ [16, Definition 3.4] (see also [1, Definition 3.2.8]). Some other valid hypotheses are discussed in [16, Remark 3.7]. We have decided to use the less technical hypotheses of amenability of G here, and to leave the more technical, but weaker, hypotheses for the interested reader to sort out as needed.

Even though Theorem 2.1 involves only groupoid C^*-algebras, our techniques use the theory of groupoid dynamical systems and their crossed products. For these, we will employ the notation and terminology from [10, Section 4]. In particular, our treatment of direct integrals comes from [8] (which was, in turn, motivated by [13]), and we suggest [19, Appendix F] as a reference. We need Renault's disintegration theorem [15, Proposition 4.2] for representations R of $C^*(G)$. For the statement, notation and basics for this result, we suggest [10, Section 7]. The disintegration result implies that R is the integrated form of a unitary representation $(\mu, G^{(0)} * \mathcal{H}, V)$ of G consisting of a quasi-invariant measure μ on $G^{(0)}$, a Borel Hilbert bundle $G^{(0)} * \mathcal{H}$ and a family $V = \{V_\gamma : \mathcal{H}(s(\gamma)) \to \mathcal{H}(r(\gamma))\}$ of unitaries so that

$$\hat{V}(\gamma) = (r(\gamma), V_\gamma, s(\gamma))$$

defines a groupoid homomorphism $\hat{V} : G \to \operatorname{Iso}(G^{(0)} * \mathcal{H})$.

The proof of Theorem 2.1 occupies the entire next section. Since the proof is a bit involved, we give a brief overview here in the hope that it will motivate some of the efforts in the next section. (The basic outline follows the proof of the Gootman-Rosenberg-Sauvageot Theorem as proved in [19, Chapter 9].)

We start by fixing $K \in \operatorname{Prim} C^*(G)$ and letting R be an irreducible representation such that $\ker R = K$. We assume that R is the integrated form of a unitary representation $(\mu, G^{(0)} * \mathcal{H}, V)$. Since V defines via restriction a representation r_u of each stability group $G(u) = \{\gamma \in G : r(\gamma) = u = s(\gamma)\}$, and since we can view each r_u as a representation of the C^*-algebra $C^*(\Sigma)$ of the group bundle Σ

associated to the collection $\Sigma^{(0)}$ of closed subgroups of G, we can form the direct integral representation

$$r := \int_{G^{(0)}}^{\oplus} r_u \, d\mu(u)$$

of $C^*(\Sigma)$. We call r the restriction of R to the isotropy groups of G.

A key step is to observe that (r, V) is a covariant representation of a groupoid dynamical system $(C^*(\Sigma), G, \alpha)$ for a natural action α. Then we can form the representation $L'' = r \rtimes V$ of $C^*(\Sigma) \rtimes_\alpha G$. This allows us to invoke Renault's impressive [16, Theorem 2.2] which is a groupoid equivariant version of Effros's ideal center decomposition theorem from [3] (for more on Effros's result, see [19, Appendix G]). This allows us to show that r is equivalent to a representation

$$\tilde{r} := \int_{\operatorname{Prim} C^*(\Sigma)}^{\oplus} \tilde{r}_P \, d\nu(P),$$

where each $\tilde{r}_P$ has kernel P, and ν is a measure on $\operatorname{Prim} C^*(\Sigma)$. Moreover, $\operatorname{Prim} C^*(\Sigma)$ is a right G-space for the action of G induced by α, and [16, Theorem 2.2] implies that ν is quasi-invariant when $\operatorname{Prim} C^*(\Sigma)$ is viewed as the unit space of the transformation groupoid $\underline{G} = \operatorname{Prim} C^*(\Sigma) * G$. (Although $\underline{G}$ is not a locally compact groupoid, it is a Borel groupoid with a Borel Haar system so the definition of quasi-invariant makes perfectly good sense.) We then need to work a bit to see that ν is also ergodic with respect to the G-action on $\operatorname{Prim} C^*(\Sigma)$.

We then define an induced representation $\operatorname{ind} \tilde{r}$ of $C^*(G)$. As essential component of the proof is Proposition 3.14 where we use the quasi-invariance and ergodicity of ν to show that the kernel of $\operatorname{ind} \tilde{r}$ is an induced primitive ideal. This is a generalization of Sauvageot's [17, Lemma 5.4] where the corresponding result for transformations groups is proved. Then the final step in our proof is to observe that $\operatorname{ind} \tilde{r}$ is equivalent to the induced representation $\operatorname{Ind} \hat{L}$ used by Renault in [16]. Then we can invoke the deep results in [16] to show, when G is suitably amenable, that

$$K = \ker R = \ker(\operatorname{Ind} \hat{L}).$$

Since $\ker(\operatorname{ind} \tilde{r}) = \ker(\operatorname{Ind} \hat{L})$ and since $\ker(\operatorname{ind} \tilde{r})$ is induced, this shows K is induced and completes the proof.

3. THE PROOF OF THE MAIN THEOREM

In this section we present the details of the proof of Theorem 2.1. As in the statement of the theorem, G will always denote a second countable locally compact Hausdorff groupoid endowed with a Haar system $\{\lambda^u\}_{u \in G^{(0)}}$.

Let K be a primitive ideal in $\operatorname{Prim} C^*(G)$. Using Renault's disintegration theorem, we can find an irreducible representation R such that $\ker R = K$ *and* such that R is the integrated form of a representation $(\mu, G^{(0)} * \mathcal{H}, V)$ of G.

We let $\Sigma^{(0)}$ be the space of closed subgroups of G equipped with the Fell topology whose basic open sets are of the form

$$\begin{aligned}\mathcal{U}(K; U_1, \dots, U_n)\\ &= \{H \in \Sigma^{(0)} : H \cap K = \emptyset \text{ and } H \cap U_i \neq \emptyset \text{ for } i = 1, 2, \dots, n\},\end{aligned}$$

where $K \subset G$ is compact and each $U_i \subset G$ is open (cf. [19, Appendix H.1]). Although $\Sigma^{(0)}$ need not be compact as in the group case, $\Sigma^{(0)} \cup \{\emptyset\}$ is compact in the space of closed subsets of G. Hence, $\Sigma^{(0)}$ is locally compact Hausdorff. Furthermore the map $p : \Sigma^{(0)} \to G^{(0)}$ given by $p(H) = u$ if $r(H) = \{u\} = s(H)$ is continuous, and if $K \subset G^{(0)}$ is compact, then $p^{-1}(K) \cap \Sigma^{(0)}$ is compact. That is, $\Sigma^{(0)}$ is conditionally compact over $G^{(0)}$ [16, Section 1]. We let Σ be the associated group bundle over $\Sigma^{(0)}$:

$$\Sigma = \{(u, H, \sigma) : u = p(H) \text{ and } \gamma \in H\}.$$

(The elements of Σ have been written slightly redundantly to make some of the subsequent computations easier to follow.) Notice that if $(u, H, \sigma) \in \Sigma$, then $H \subset G(u)$. By [16, Corollary 1.4], there is a continuous Haar system $\{\beta^H\}_{H \in \Sigma^{(0)}}$ for Σ.

We want to define an action α of G on $C^*(\Sigma)$ so that $(C^*(\Sigma), G, \alpha)$ is a groupoid dynamical system. We start by showing that $C^*(\Sigma)$ is a $C_0(G^{(0)})$-algebra (cf. [19, Definition C.1]). For this, the following variation on [16, Lemma 1.6] will be helpful.

Lemma 3.1. *Let $G * \Sigma^{(0)} = \{(\sigma, H) \in G \times \Sigma^{(0)} : s(\sigma) = p(H)\}$. Then there is a continuous map $\omega : G * \Sigma^{(0)} \to (0, \infty)$ such that*

$$\int_H f(\sigma\gamma\sigma^{-1})\, d\beta^H(\gamma) = \omega(\sigma, H) \int_{\sigma \cdot H} f(\gamma)\, d\beta^{\sigma \cdot H}(\gamma) \quad \text{for all } f \in C_c(G). \tag{3.1}$$

Furthermore, for all σ, $\tau \in G$ and $H \in \Sigma^{(0)}$, we have

$$\omega(\sigma\tau, H) = \omega(\tau, H)\omega(\sigma, \tau \cdot H) \quad \text{and} \quad \omega(\sigma, H)^{-1} = \omega(\sigma^{-1}, \sigma \cdot H). \tag{3.2}$$

Sketch of the Proof. The existence of $\omega(\sigma, H)$ follows from the uniqueness of the Haar measure on $\sigma \cdot H$. The fact that ω is continuous follows from the fact that both integrals in (3.1) are continuous with respect to (σ, H). Equation (3.2) is a straightforward computation. ◻

For $u \in G^{(0)}$, let $\Sigma_{G(u)}$ the compact Hausdorff space of subgroups of $G(u)$. Let $C^*(\Sigma_{G(u)})$ be the groupoid C^*-algebra of the corresponding group bundle. Thus if $\Sigma_{G(u)} * G = \{(H, \gamma) \in \Sigma^{(0)} \times G : \gamma \in H\}$, then $C^*(\Sigma_{G(u)})$ is the completion of $C_c(\Sigma_{G(u)} * G)$ in the obvious universal norm for the $*$-algebra structure given by

$$f * g(H, \gamma) = \int_H f(H, \eta) g(H, \eta^{-1}\gamma)\, \mathrm{d}\beta^H(\eta) \quad \text{and} \quad f^*(H, \gamma) = f(H, \gamma^{-1})^*.$$

Since the restriction map, $\kappa_u : C_c(\Sigma) \to C_c(\Sigma_{G(u)})$ is surjective (by [19, Lemma 8.54]), κ_u extends to a homomorphism of $C^*(\Sigma)$ onto $C^*(\Sigma_{G(u)})$.

Remark 3.2. Notice that $C^*(\Sigma_{G(u)})$ is (isomorphic to) Fell's subgroup C^*-algebra as originally defined in [5] (or as a special case of [19, Section 8.4]). It is important to note that, since we are treating $\Sigma_{G(u)} * G$ as a groupoid, there are no modular functions in the formula above for the adjoint in contrast to the definitions in [5] or [19].

Lemma 3.3. *The groupoid C^*-algebra $C^*(\Sigma)$ is a $C_0(G^{(0)})$-algebra. Moreover, the fiber $C^*(\Sigma)(u)$ over u is isomorphic to $C^*(\Sigma_{G(u)})$.*

Proof. The groupoid C^*-algebra $C^*(\Sigma)$ is clearly a $C_0(\Sigma^{(0)})$-algebra, and as in the proof of [19, Proposition 8.55], it is not hard to check that the fibre $C^*(\Sigma)(H)$ over H is isomorphic to $C^*(H)$. In particular, the restriction map $\iota_H : C_c(\Sigma) \to C_c(H)$ is surjective and extends to a homomorphism of $C^*(\Sigma)$ onto $C^*(H)$.

By composing functions on $G^{(0)}$ with p, we see that $C^*(\Sigma)$ is also a $C_0(G^{(0)})$-algebra. Let $u \in G^{(0)}$ and let I_u be the ideal of $C^*(\Sigma)$ spanned by $C_{0,u}(G^{(0)}) \cdot C_c(\Sigma)$, where $C_{0,u}(G^{(0)})$ consists of the functions f in $C_0(G^{(0)})$ such that $f(u) = 0$. Then $C^*(\Sigma)(u) = C^*(\Sigma)/I_u$. Clearly $I_u \subset \ker \kappa_u$. To show that $C^*(\Sigma)(u)$ is isomorphic to $C^*(\Sigma_{G(u)})$ it is enough to prove that $I_u \supset \ker \kappa_u$.

Let L be a representation of $C^*(\Sigma)$ such that $I_u \subset \ker L$. An approximation argument shows that if $f \in C_c(\Sigma)$ is such that $f(u, H, \gamma) = 0$ for all $H \in G(u)$ and $\gamma \in H$, then $f \in \ker L$. Therefore if $\varphi \in C_0(\Sigma)$ is such that $\varphi(H) = 1$ for all $H \in \Sigma_{G(u)}$, then $L(f) = L(\varphi \cdot f)$.

We can view $\Sigma_{G(u)}$ as a *compact* subset of Σ. Since

$$H \mapsto \int_H f(\gamma, H)\, \mathrm{d}\beta^H(\gamma)$$

is continuous on Σ, for any $\varepsilon > 0$ we can find $\varphi \in C_0(\Sigma)$ such that $\varphi(H) = 1$ for all $H \in \Sigma_{G(u)}$ and such that

$$\|\varphi \cdot f\|_{C^*(\Sigma)} \le \sup_{H \in \Sigma_{G(u)}} \|\iota_H(f)\|_1 + \varepsilon,$$

where $\iota_H : C_c(\Sigma) \to C_c(H)$ is the restriction map. It follows that

$$\|L(f)\| \le \sup_{H \in \Sigma_{G(u)}} \|\iota_H(f)\|_1 \le \|\kappa_u(f)\|_I,$$

where $\|\cdot\|_I$ is the I-norm on $C_c(\Sigma_{G(u)} * G) \subset C^*(\Sigma_{G(u)})$. Thus we can define a $\|\cdot\|_I$-decreasing representation L' of $C_c(\Sigma_{G(u)})$ by $L'(\kappa_u(f)) := L(f)$. Since L' must be norm decreasing for the C^*-norm, we have

$$\|L(f)\| \le \|\kappa_u(f)\|_{C^*(\Sigma_{G(u)})},$$

and $\ker \kappa_u \subset \ker L$. Since L is any representation with I_u in its kernel, we have $\ker \kappa_u \subset I_u$. □

To define an action α of G on $C^*(\Sigma)$ using [10, Definition 4.1], we first define

$$\alpha_\eta : C^*(\Sigma)(s(\eta)) \to C^*(\Sigma)(r(\eta))$$

at the level of functions by

$$\alpha_\eta(F)(r(\gamma), H, \gamma) := \omega(\eta^{-1}, H)^{-1} F(s(\eta), \eta^{-1}\cdot H, \eta^{-1}\gamma\eta). \tag{3.3}$$

Then we compute that

$$\begin{aligned}\int_H \alpha_\eta(F)(r(\eta), H, \gamma)\, d\beta^H(\gamma) \\ &= \omega(\eta^{-1}, H)^{-1}\int_H F(s(\eta), \eta^{-1}\cdot H, \eta^{-1}\gamma\eta)\, d\beta^H(\gamma) \\ &= \int_{\eta^{-1}\cdot H} F(s(\eta), \eta^{-1}\cdot H, \gamma)\, d\beta^{\eta^{-1}\cdot H}(\gamma).\end{aligned} \tag{3.4}$$

Lemma 3.4. *The triple $(C^*(\Sigma), G, \alpha)$ is a groupoid dynamical system.*

Proof. The preceding discussion shows that α_η is isometric for the I-norm, and hence defines an isomorphism. The fact that $\alpha_{\gamma\delta} = \alpha_\gamma \circ \alpha_\delta$ is clear by equation (3.3). To see that $\alpha = \{\alpha_\eta\}_{\eta\in G}$ is continuous is a bit messy. We'll use the criteria from [10, Lemma 4.3] and show that there is a $C_0(G)$-linear isomorphism

$$\alpha : r^*C^*(\Sigma) \to s^*C^*(\Sigma)$$

which induces the α_η on the fibres. It is not hard to establish that the pull-back $r^*C^*(\Sigma)$ is ($C_0(G)$-isomorphic to) the C^*-algebra of the group bundle $G *_r \Sigma^{(0)} * G = \{(\eta, H, \gamma) : \gamma \in H \subset G(r(\eta))\}$, and similarly for $s^*C^*(\Sigma)$. Then we can define

$$\alpha : C_c(G *_r *\Sigma^{(0)} * G) \to C_c(G *_s *\Sigma^{(0)} * G)$$

by

$$\alpha(f)(\eta, H, \gamma) = \omega(\eta^{-1}, H)^{-1} f(\eta, \eta^{-1}\cdot H, \eta^{-1}\gamma\eta).$$

Then α is isometric with respect to the appropriate I-norms and therefore extends to a $C_0(G)$-linear isomorphism which induces the α_η as required. □

3.1. Restriction to the stability groups. We maintain the set-up that R is an irreducible representation with $\ker R = K \in \operatorname{Prim} C^*(\Sigma)$, and that R is the integrated form of a representation $(\mu, G^{(0)} * \mathcal{H}, V)$ of G. Note that $C^*(\Sigma) = \Gamma_0(G^{(0)}; S)$ for an upper semicontinuous C^*-bundle $p_S : S \to G^{(0)}$ (as in [19, Theorem C.26]). Since $u \mapsto G(u)$ is Borel [16, Lemma 1.5], we can define the *restriction of* R *to the isotropy groups of* G to be the representation r of $C^*(\Sigma)$ on $L^2(G^{(0)} * \mathcal{H}, \mu)$ given by

$$r(F)h(u) := \int_{G(u)} F(u, G(u), \gamma) V_\gamma h(u) \Delta_{G(u)}(\gamma)^{-1/2} \, \mathrm{d}\beta^{G(u)}(\gamma), \tag{3.5}$$

where $\Delta_{G(u)}$ is the modular function on $G(u)$ (see Remark 3.5 below). It may be helpful to notice that r is the direct integral

$$r = \int_{G^{(0)}}^{\oplus} r_u \, \mathrm{d}\mu(u), \tag{3.6}$$

where r_u is the composition of the representation of $C^*(G(u))$ given by $V|_{G(u)}$ with the quotient map κ_u of $C^*(\Sigma)$ onto $C^*(G(u))$. (We will also write r_u for the representation of $C^*(G(u))$.)

Remark 3.5. Some care is necessary when applying r_u to a function in $F \in C_c(\Sigma)$—or for that matter, $C_c(G(u))$. Since Σ is a groupoid, there is no modular function in the formula for the adjoint; instead, it must appear in the integrated form of representations in order that they be $*$-preserving. This "explains" the appearance of the group modular functions in (3.5). In fact, $\delta'(u, \gamma) = \Delta_{G(u)}(\gamma)$ is the Radon-Nikodym derivative of $\mu \circ \beta$ with respect to $\mu \circ \beta^{-1}$ associated to μ considered as a *quasi-invariant* measure on $G^{(0)}$ viewed as the unit space of the Borel groupoid group bundle $G' = \{(u, \gamma) : \gamma \in G(u)\}$. This observation will be important at the end of Section 3.2.

Lemma 3.6. *The tuple* $(r, \mu, G^{(0)} * \mathcal{H}, V)$ *is a covariant representation of* $(C^*(\Sigma), G, \alpha)$ *[10, Definition 7.9]. In fact,* $V_\gamma r_{s(\gamma)} = (r_{r(\gamma)} \circ \alpha_\gamma) V_\gamma$ *for* all $\gamma \in G$.

Proof. We compute as follows. Fix $\gamma \in G$ with $s(\gamma) = v$ and $r(\gamma) = u$. Then

$$\begin{aligned}
&V_\gamma r_v(F) \\
&= V_\gamma \int_{G(v)} F(v, G(v), \eta) V_\eta \Delta_{G(v)}(\eta)^{-1/2} \, \mathrm{d}\beta^{G(v)}(\eta) \\
&= \int_{G(v)} F(v, G(v), \eta) V_{\gamma\eta} \Delta_{G(v)}(\eta)^{-1/2} \Delta_{G(v)}(\eta)^{-1/2} \, \mathrm{d}\beta^{G(v)}(\eta) \\
&= \omega(\gamma, G(v)) \int_{\gamma \cdot G(v)} F(v, G(v), \gamma^{-1}\eta\gamma) V_{\eta\gamma} \Delta_{G(v)}(\gamma^{-1}\eta\gamma) \, \mathrm{d}\beta^{\gamma \cdot G(v)}(\eta)
\end{aligned}$$

which, since $\gamma \cdot G(v) = G(u)$ and $\Delta_{G(v)}(\gamma^{-1}\eta\gamma) = \Delta_{G(u)}(\eta)$, is

$$\begin{aligned} &= \int_{G(u)} \omega(\gamma, \gamma^{-1} \cdot G(u)) F(v, \gamma^{-1} \cdot G(u), \gamma^{-1}\eta\gamma) V_\eta \Delta_{G(u)}(\eta)^{-1/2} \, d\beta^{G(u)} V_\gamma \\ &= \int_{G(u)} \alpha_\gamma(F)(u, G(u), \eta) V_\eta \Delta_{G(u)}(\eta)^{-1/2} \, d\beta^{G(u)}(\eta) V_\gamma \\ &= r_u(\alpha_\gamma(F)) V_\gamma. \end{aligned}$$

The result follows. ◻

In view of Lemma 3.6, we can let $L'' := r \rtimes V$ be the representation of the groupoid crossed product $C^*(\Sigma) \rtimes_\alpha G$ which is the integrated form of $(r, \mu, G^{(0)} * \mathcal{H}, V)$ (see [10, Proposition 7.11]). If δ is the Radon Nikodym derivative of $\mu \circ \lambda$ with respect to $\mu \circ \lambda^{-1}$, then for each $f \in \Gamma_c(G; r^*S)$

$$(L''(f)h \mid k) = \int_{G^{(0)}} \int_G (r_u(f(\gamma)) V_\gamma h(s(\gamma)) \mid k(u)) \delta(\gamma)^{-1/2} \, d\lambda^u(\gamma) \, d\mu(u).$$

Now we want to form Effros's ideal center decomposition of r following [16, Theorem 2.2].[2] Let $\sigma : \operatorname{Prim} C^*(\Sigma) \to G^{(0)}$ be the continuous map induced by the $C_0(G^{(0)})$-structure on $C^*(\Sigma)$ [19, Proposition C.5]. As in the discussion preceding [16, Proposition 1.14], there is a continuous G-action on $\operatorname{Prim} C^*(\Sigma)$, equipped with its Polish *regularized* topology, with respect to σ; that is, there is a continuous map $(P, \gamma) \mapsto P \cdot \gamma$ from

$$\operatorname{Prim} C^*(\Sigma) * G = \{(P, \gamma) : \sigma(P) = r(\gamma)\}$$

to $\operatorname{Prim} C^*(\Sigma)$.[3] Then, as in the paragraph following the proof of [16, Proposition 1.14], we can form the transformation groupoid

$$\underline{G} := \operatorname{Prim} C^*(\Sigma) * G,$$

where

$$\begin{aligned} r(P, \gamma) &= P, & s(P, \gamma) &= P \cdot \gamma, \\ (P, \gamma)(P \cdot \gamma, \eta) &= (P, \gamma\eta), & (P, \gamma)^{-1} &= (P \cdot \gamma, \gamma^{-1}). \end{aligned}$$

[2] Formally, Renault's proofs need to be modified to deal with upper semicontinuous C^*-bundles, but this is straightforward.

[3] Notice that in many treatments of groupoid actions on spaces, the structure map σ is assumed to be open as well as continuous. In this case, we have only that σ is continuous. Since $\operatorname{Prim} A$ has the regularized topology, there is no reason to suspect that σ need be open even if S were a continuous C^*-bundle in the first place.

With respect to the regularized topology on $\operatorname{Prim} C^*(\Sigma)$, $\underline{G}$ is what Renault calls in [16] *locally conditionally compact*—the important thing is that it is a standard Borel groupoid, and that $\underline{\lambda}^P = \varepsilon_P \times \lambda^{\sigma(P)}$ is a continuous Haar system for $\underline{G}$ [16, p. 12]. Using this structure, we want to construct an *covariant ideal center decomposition* of L'' in analogy with [19, Appendix G.2]. The idea is to use a decomposition theorem of Effros's [3] to decompose r into homogeneous representations in an equivariant way (cf. [19, Theorem C.22]). Recall that a representation π of a C^*-algebra A is homogeneous if $\ker \pi = \ker \pi^E$ for any nonzero projection $E \in \pi(A)'$. (Here π^E is the subrepresentation of π corresponding to E.) It was Sauvageot who first noticed the importance of Effros's ideal center decomposition for the solution of the Effros-Hahn conjecture. A key feature for us is that if A is separable and π is homogeneous, then $\ker \pi$ is primitive [19, Corollary G.9]. Renault provides the decomposition result we need in [16, Theorem 2.2]. The essential features of his result are as follows: L'' is equivalent to a representation L' on $L^2(\operatorname{Prim} C^*(\Sigma) * \mathcal{K}, \nu)$ where ν is a quasi-invariant measure on $\underline{G}^{(0)} = \operatorname{Prim} C^*(\Sigma)$, and $\operatorname{Prim} C^*(\Sigma) * \mathcal{K}$ is a Borel Hilbert bundle over $\operatorname{Prim} C^*(\Sigma)$. To define L', Renault must produce a ν-conull set $U \subset \operatorname{Prim} C^*(\Sigma)$ and a Borel homomorphism

$$\hat{L}' : \underline{G}|_U \to \operatorname{Iso}\left(\operatorname{Prim} C^*(\Sigma) * \mathcal{K}\right)$$

of the form $\hat{L}'(P,\gamma) = (P, L_{(P,\gamma)}, P \cdot \gamma)$, and for each $P \in U$, homogeneous representations $\tilde{r}_P$ of $C^*(\Sigma)$ with $\ker \tilde{r}_P = P$ such that for all $F \in C^*(\Sigma)$

$$P \mapsto \tilde{r}_P(F) \quad \text{is Borel, and} \tag{3.7}$$

$$L(P,\gamma)\tilde{r}_{P\cdot\gamma}(F) = \tilde{r}_P(\alpha_\gamma(F))L(P,\gamma) \quad \text{for all } (P,\gamma) \in \underline{G}|_U. \tag{3.8}$$

Since $\ker \tilde{r}_P = P$, we can always view $\tilde{r}_P$ as a representation of the fibre $C^*(\Sigma)(\sigma(P))$. Therefore, if $f \in \Gamma_c(G; r^*S)$, then $(\gamma, P) \mapsto \tilde{r}_P(f(\gamma))$ is well-defined and Borel on the set of (P,γ) such that $\sigma(P) = r(\gamma)$. (Recall that sections of the form $\gamma \mapsto \varphi(\gamma)a(r(\gamma))$ are dense in $\Gamma_c(G; r^*S)$ in the inductive limit topology.) The primary conclusion of [16, Theorem 2.2] is that L'' is equivalent to the representation defined by

$$L'(f)h(P) = \int_G \tilde{r}_P(f(\gamma))L(P,\gamma)\Delta(P,\gamma)^{-1/2}h(P\cdot\gamma)\, d\lambda^{\sigma(P)}(\gamma),$$

where Δ is the Radon-Nikodym derivative of $\nu \circ \underline{\lambda}^{-1}$ with respect to $\nu \circ \underline{\lambda}$. Then L' is what we meant by a covariant ideal center decomposition of L'' (see Remark 3.8 below and compare with [19, Proposition G.24 and Lemma 9.9]).

Notice that (3.8) implies that

$$\tilde{r}_{P\cdot\gamma} \cong \gamma \cdot \tilde{r}_P \quad \text{for all } (P,\gamma) \in \underline{G}|_U. \tag{3.9}$$

On the other hand, in view of (3.7), we can form the direct integral representation of $C^*(\Sigma)$ given by

$$\tilde{r} := \int_{\operatorname{Prim} C^*(\Sigma)}^{\oplus} \tilde{r}_P \, \mathrm{d}\nu(P). \tag{3.10}$$

Lemma 3.7. *Let R be an irreducible representation of $C^*(G)$ and suppose that r and $\tilde{r}$ are the representations of $C^*(\Sigma)$ defined in (3.6) and (3.10), respectively. Then r and $\tilde{r}$ are equivalent.*

Remark 3.8. As a consequence of Lemma 3.7, we note that $\tilde{r}$ is an ideal center decomposition of r as defined in [19, Definition G.18]. This justifies the terminology used above.

Proof of Lemma 3.7. If $a \in C^*(\Sigma) = \Gamma_0(G^{(0)}; S)$ and $f \in \Gamma_c(G; r^*S)$, then we can define $a \cdot f \in \Gamma_c(G; r^*S)$ by

$$a \cdot f(\gamma) := a(r(\gamma))f(\gamma).$$

Then, viewing $\tilde{r}_P$ as a representation of the fibre $C^*(\Sigma)(\sigma(P))$, we have $\tilde{r}_P(a \cdot f(\gamma)) = \tilde{r}_P(a)\tilde{r}_P(f(\gamma))$. Thus,

$$\begin{aligned} L'(a \cdot f)h(P) &= \int_G \tilde{r}_P(a \cdot f(\gamma))L(P,\gamma)\Delta(P,\gamma)^{-1/2}h(P\cdot\gamma)\,\mathrm{d}\lambda^{\sigma(P)}(\gamma) \\ &= \tilde{r}_P(a)\int_G \tilde{r}_P(f(\gamma))L(P,\gamma)\Delta(P,\gamma)^{-1/2}h(P\cdot\gamma)\,\mathrm{d}\lambda^{\sigma(P)}(\gamma) \\ &= \tilde{r}_P(a)L'(f)h(P). \end{aligned}$$

That is, $L'(a\cdot f) = \tilde{r}(a)L'(f)$. Similarly, $L''(a \cdot f) = r(a)L''(f)$.

Now let $M^R : L^2(G^{(0)} * \mathcal{H}, \mu) \to L^2(\operatorname{Prim} C^*(\Sigma), \nu)$ be a unitary isomorphism intertwining L'' and L'. Then we compute that for all $a \in C_c(\Sigma)$ and $f \in \Gamma_c(G; r^*S)$ we have

$$\begin{aligned} M^R r(a)L''(f)h &= M^R L''(a\cdot f)h \\ &= L'(a\cdot f)M^R h \\ &= \tilde{r}(a)L'(f)M^R h \\ &= \tilde{r}(a)M^R L''(f)h. \end{aligned}$$

Since L'' is nondegenerate, $M^R r(a) = \tilde{r}(a)M^R$ for all $a \in C_c(\Sigma)$. Thus r and $\tilde{r}$ are equivalent as claimed. □

A subset $U \subset \operatorname{Prim} C^*(\Sigma)$ is $\underline{G}$-invariant if $U \cdot \underline{G} \subset U$. If ν is a quasi-invariant measure on $\operatorname{Prim} C^*(\Sigma)$ and if V is ν-conull, then $\underline{G}|_V$ is conull with respect to $\nu \circ \underline{\lambda}$. We say that $U \subset \operatorname{Prim} C^*(\Sigma)$ is ν-*essentially invariant* if there is a ν-conull set V such that $U \cdot \underline{G}|_V \subset U$. Notice that if U is ν-essentially invariant, then $\varphi = \mathbb{1}_U$ is *invariant* in the sense that $\varphi \circ s = \varphi \circ r$ for $\nu \circ \underline{\lambda}$-almost all (P, γ). In general, a quasi-invariant measure ν is called *ergodic* for the action of G on $\operatorname{Prim} C^*(\Sigma)$ if every Borel function φ on $\operatorname{Prim} C^*(\Sigma)$ which is invariant in the above sense is constant ν-almost everywhere (see [12, p. 274]). It is not hard to see that it suffices to consider φ which are characteristic functions of a Borel set. Furthermore, it follows from the above and [12, Lemma 5.1] or [8, Lemma 4.9] that $\varphi = \mathbb{1}_U$ is invariant if and only if U is ν-essentially invariant.[4]

Proposition 3.9. *Let G be a second countable Hausdorff groupoid G with Haar system $\{\lambda^u\}_{u\in G^{(0)}}$. Assume that R is an irreducible representation of $C^*(G)$. Let r be the restriction of R to the isotropy groups of G defined by* (3.6)*. Then the quasi-invariant measure ν on* $\operatorname{Prim} C^*(\Sigma)$ *in the ideal center decomposition* (3.10) *of r is ergodic with respect to the action of G on* $\operatorname{Prim} C^*(\Sigma)$.

For the proof, it will be convenient to make the following observation (compare with the first part of the proof of [10, Theorem 7.12]).

Lemma 3.10. *If $f \in \Gamma_c(G; r^*S)$ and $\varphi \in C_c(G)$, then define $\varphi \cdot f \in \Gamma_c(G; r^*S)$ by*

$$\varphi \cdot f(\gamma) = \int_G \varphi(\eta)\alpha_\eta(f(\eta^{-1}\gamma))\, d\lambda^{r(\gamma)}(\eta).$$

Then $L''(\varphi \cdot f) = R(\varphi)L''(f)$.

Proof. We simply compute using Fubini's Theorem as follows:

$$\begin{aligned} L''(\varphi \cdot f)h(u) &= \int_G r_u(\varphi \cdot f(\gamma))V_\gamma h(r(\gamma))\delta(\gamma)^{-1/2}\, d\lambda^u(\gamma) \\ &= \int_G \int_G \varphi(\eta) r_u(\alpha_\eta(f(\eta^{-1}\gamma)))V_\gamma h(s(\gamma))\delta(\gamma)^{-1/2}\, d\lambda^u(\gamma)\, d\lambda^u(\eta) \end{aligned}$$

which, after sending $\gamma \mapsto \eta\gamma$, is

$$= \int_G \int_G \varphi(\eta) r_u(\alpha_\eta(f(\gamma)))V_{\eta\gamma} h(s(\gamma))\delta(\eta\gamma)^{-1/2}\, d\lambda^{s(\eta)}(\gamma)\, d\lambda^u(\eta)$$

which, in view of Lemma 3.6, is

$$\begin{aligned} &= \int_G \varphi(\eta)V_\eta\Big(\int_G r_u(f(\gamma))V_\gamma h(s(\gamma))\delta(\gamma)^{-1/2}\, d\lambda^{s(\eta)}(\gamma)\Big)\delta(\eta)^{-1/2}\, d\lambda^u(\eta) \\ &= \int_G \varphi(\eta)V_\eta L''(f)h(s(\eta))\delta(\eta)^{-1/2}\, d\lambda^u(\eta) \\ &= R(\varphi)L''(f)h(u). \end{aligned}$$

□

[4] We thank the referee for pointing out the proper relationship between "ν-essentially invariant" sets and the usual notion of ergodicity for measured groupoids as laid out in [12].

Proof of Proposition 3.9. Recall that the diagonal operators are the multiplication operators T_φ for φ a bounded Borel function on $\operatorname{Prim} C^*(\Sigma)$ (see [19, Definition F.13]). Let $B \subset \operatorname{Prim} C^*(\Sigma)$ be a $\underline{G}|_V$-invariant Borel subset for a ν-conull set $V \subset \operatorname{Prim} C^*(\Sigma)$, and let $\varphi = \mathbb{1}_B$. Then φ is a bounded Borel function on $\operatorname{Prim} C^*(\Sigma)$ and we can let $E = T_\varphi$ be the corresponding diagonal operator on $L^2(\operatorname{Prim} C^*(\Sigma) * \mathcal{K}, \nu)$. It will suffice to show that E is either the identity or the zero operator. Since for ν-almost all P, $\varphi(P \cdot \gamma) = \varphi(P)$ for $\lambda^{\sigma(P)}$-almost all γ, it is clear from the definition of L' that E commutes with $L'(f)$ for all $f \in \Gamma_c(G; r^*S)$. Thus $E'' := (M^R)^{-1} E M^R$ commutes with $L''(f)$ for all $f \in \Gamma_c(G; r^*S)$. But if $\varphi \in C_c(G)$, then using Lemma 3.10, we have

$$\begin{aligned} R(\varphi)E''L''(f)h &= R(\varphi)L''(f)E''h \\ &= L''(\varphi \cdot f)E''h \\ &= E''R(\varphi)L''(f)h \end{aligned}$$

for all $\varphi \in C_c(G)$, $f \in \Gamma_c(G; r^*S)$ and $h \in L^2(G^{(0)} * \mathcal{H}, \mu)$. Since L'' is nondegenerate, E'' commutes with every $R(\varphi)$. Since R is assumed irreducible, E'', and therefore E, must be trivial. ◻

Let $C^b(G^{(0)})$ and $\mathcal{B}^b(G^{(0)})$ denote, respectively, the bounded continuous and bounded Borel functions on $G^{(0)}$. If $\varphi \in \mathcal{B}^b(G^{(0)})$, then we will write T_φ and $T^\Sigma_{\varphi\circ\sigma}$ for the corresponding diagonal operators in $L^2(G^{(0)} * \mathcal{H}, \mu)$ and $L^2(\operatorname{Prim} C^*(\Sigma) * \mathcal{K}, \nu)$, respectively. Notice that if $\{\varphi_i\} \subset \mathcal{B}^b(G^{(0)})$ is a bounded sequence converging to $\varphi \in \mathcal{B}^b(G^{(0)})$ μ-almost everywhere, then by the dominated convergence theorem, $T_{\varphi_i} \to T_\varphi$ in the strong operator topology.

Lemma 3.11. *The isomorphism M^R which implements the equivalence between r and $\tilde{r}$ intertwines the diagonal operators on*

$$L^2(G^{(0)} * \mathcal{H}, \mu) \quad \text{and} \quad L^2(\operatorname{Prim} C^*(\Sigma) * \mathcal{K}, \nu).$$

In fact, we have

$$M^R T_\varphi = T^\Sigma_{\varphi\circ\sigma} M^R \quad \text{for all } \varphi \in \mathcal{B}^b(G^{(0)}). \tag{3.11}$$

Proof. If $\varphi \in C_0(G^{(0)})$ and $F \in C_c(\Sigma)$, then [19, Proposition C.5] (and the discussion preceding it) implies that $\tilde{r}_P(\varphi \cdot F) = \varphi(\sigma(P))\tilde{r}_P(F)$, and it is not hard to see that this formula still holds when $\varphi \in C^b(G^{(0)})$.[5] Thus if $\varphi \in C^b(G^{(0)})$ and $F \in C_c(\Sigma)$, then

[5] The action of $C_0(G^{(0)})$ is given by a *nondegenerate* homomorphism of $C_0(G^{(0)})$ into the center of $M(C^*(\Sigma))$ and so extends to $C^b(G^{(0)}) = M(C_0(G^{(0)}))$.

$$\begin{aligned} M^R T_\varphi r(F) &= M^R r(\varphi \cdot F) \\ &= \tilde{r}(\varphi \cdot F) M^R \\ &= T^{\Sigma}_{\varphi\circ\sigma} \tilde{r}(F) M^R \\ &= T^{\Sigma}_{\varphi\circ\sigma} M^R r(F). \end{aligned}$$

Since r is nondegenerate, we have shown that (3.11) holds for all $\varphi \in C^b(G^{(0)})$.

Now suppose that M is a μ-null set. We claim that $\sigma^{-1}(M)$ is ν-null. Since μ is a Radon measure and $G^{(0)}$ is second countable, we may as well assume that M is a G_δ subset of a compact set. But then there is a bounded sequence $\{\varphi_i\} \subset C_0^+(G^{(0)})$ such that $\varphi_i \searrow \mathbb{1}_M$ *everywhere*. Then $\varphi_i \circ \sigma \searrow \varphi \circ \sigma$ everywhere. It follows that in the strong operator topology, we have $T_{\varphi_i} \to T_{\mathbb{1}_M} = 0$ and $T^{\Sigma}_{\varphi_i\circ\sigma} \to T^{\Sigma}_{\mathbb{1}_{\sigma^{-1}(M)}}$. Since (3.11) holds for continuous functions, it follows that $T^{\Sigma}_{\mathbb{1}_{\sigma^{-1}(M)}} = 0$. That is, $\sigma^{-1}(M)$ is ν-null.

Now if $\varphi \in \mathcal{B}^b(G^{(0)})$, then we can find a bounded sequence $\{\varphi_i\} \subset C^b(G^{(0)})$ such that $\varphi_i \to \varphi$ μ-almost everywhere. In view of the previous paragraph, $\varphi_i \circ \sigma \to \varphi \circ \sigma$ ν-almost everywhere. Therefore $T_{\varphi_i} \to T_\varphi$ and $T^{\Sigma}_{\varphi_i\circ\sigma} \to T^{\Sigma}_{\varphi\circ\sigma}$ in the strong operator topology, and since (3.11) holds for each φ_i, it follows that (3.11) holds for all φ. □

As we saw in the previous proof, $\sigma_*\nu \ll \mu$, where $\sigma_*\nu$ is the push-forward of ν under σ: $\sigma_*\nu(E) = \nu(\sigma^{-1}(E))$. Therefore, using [19, Corollary I.9], we can disintegrate ν with respect to μ. This means that there are finite measures ν_u on $\operatorname{Prim} C^*(\Sigma)$ supported in $\sigma^{-1}(u)$ such that

$$\int_{\operatorname{Prim} C^*(\Sigma)} \varphi(P)\, d\nu(P) = \int_{G^{(0)}} \int_{\operatorname{Prim} C^*(\Sigma)} \varphi(P)\, d\nu_u(P)\, d\mu(u)$$

for any bounded Borel function φ on $\operatorname{Prim} C^*(\Sigma)$. Since $P \mapsto \tilde{r}_P$ is a Borel field of representations, we can form the direct integral representation

$$\hat{r}_u := \int^{\oplus}_{\operatorname{Prim} C^*(\Sigma)} \tilde{r}_P \, d\nu_u(P) \tag{3.12}$$

on $\mathcal{V}_u := L^2(\operatorname{Prim} C^*(\Sigma) * \mathcal{K}, \nu_u)$. We can form then the Borel Hilbert bundle $G^{(0)} * \mathcal{V}$ induced from $L^2(\operatorname{Prim} C^*(\Sigma) * \mathcal{K}, \nu)$ via the disintegration of ν with respect to μ (see [19, Example F.19]). Given the fact that we can identify $L^2(\operatorname{Prim} C^*(\Sigma) * \mathcal{K}, \nu)$ with $L^2(G^{(0)} * \mathcal{V}, \mu)$, it follows that $\tilde{r}$ is equivalent to

$$\hat{r} = \int^{\oplus}_{G^{(0)}} \hat{r}_u \, d\mu(u).$$

If $\varphi \in B^b(G^{(0)})$ and T'_φ is the corresponding diagonal operator on $L^2(G^{(0)} * \mathcal{V}, \mu)$, then

$$M^R T_\varphi = T'_\varphi M^R.$$

Therefore [19, Corollary F.34] implies that r_u and $\hat{r}_u$ are equivalent for μ-almost all u.

Since $\ker \hat{r}_u$ is separable, equation (3.12) implies that there exists a ν_u-null set $N(u)$ such that

$$\ker \hat{r}_u \subset \ker \tilde{r}_P \quad \text{if } P \notin N(u).$$

Since $\operatorname{supp} \nu_u \subset \sigma^{-1}(u)$ we can rewrite this as

$$\ker \hat{r}_{\sigma(P)} \subset \ker \tilde{r}_P \tag{3.13}$$

for ν_u-almost all P and for all u. It follows that (3.13) holds for ν-almost all P. Thus off a ν-null set N, $\tilde{r}_P$ factors through $C^*(G(\sigma(P)))$.

3.2. The induced representation. We are retaining the notation and assumptions from the previous section: R is an irreducible representation of $C^*(G)$ with kernel K and r is the restriction of R to the isotropy groups of G. We have seen that if

$$\tilde{r} = \int^{\oplus}_{\operatorname{Prim} C^*(\Sigma)} \tilde{r}_P \, d\nu(P)$$

is the ideal center decomposition of r defined in (3.10), then $\tilde{r}_P$ factors through $C^*(G(\sigma(P)))$ for almost all P. Next we want to form an induced representation $\operatorname{ind} \tilde{r}$ of $C^*(G)$. First we recall some of the basics of induced representations of groupoids.

Suppose that ρ is a representation of $C^*(G(u))$. Then, using the notation from [7, Section 2], we define $\operatorname{Ind}_{G(u)}^G \rho$ to be representation of $C^*(G)$ on the completion of the $C_c(G_u) \odot \mathcal{H}_\rho$ with respect to the inner product defined on elementary tensors by

$$(\varphi \otimes h \mid \psi \otimes k) = (\rho(\langle \psi \, , \varphi \rangle_\star) h \mid k).$$

Then

$$\operatorname{Ind}_{G(u)}^G \rho(f)(\varphi \otimes h) = f * \varphi \otimes h.$$

If $\mathcal{I}(A)$ denotes the space of (closed two-sided) ideals in A, then, as in [11, Proposition 3.34 and Corollary 3.35], there is a continuous map

$$\operatorname{Ind}_{G(u)}^G : \mathcal{I}(C^*(G(u))) \to \mathcal{I}(C^*(G))$$

characterized by

$$\ker(\operatorname{Ind}_{G(u)}^G \rho) = \operatorname{Ind}_{G(u)}^G (\ker \rho).$$

Since $\operatorname{Ind}_{G(u)}^G \rho$ is irreducible if ρ is [7, Theorem 5], it follows that $\operatorname{Ind}_{G(u)}^G J \in \operatorname{Prim} C^*(G)$ if $J \in \operatorname{Prim} C^*(G(u))$. Recall that we call K an *induced primitive ideal* if K is primitive and $K = \operatorname{Ind}_{G(u)}^G J$ for some $J \in \operatorname{Prim} C^*(G(u))$.

The following is a straightforward consequence of the definitions.

Lemma 3.12. *Let ρ be a representation of $C^*(G(u))$ and let $\gamma \in G_u$. Let $\gamma \cdot \rho$ be the representation of $C^*(G(r(\gamma)))$ given by $\gamma \cdot \rho(a) := \rho(\gamma^{-1} a \gamma)$. Then $\operatorname{Ind}_{G(u)}^G \rho$ and $\operatorname{Ind}_{G(r(\gamma))}^G \gamma \cdot \rho$ are equivalent representations.*

Sketch of the Proof. Define $V : C_c(G_u) \to C_c(G_{r(\gamma)})$ by

$$V(f)(h) = \omega(\gamma, G(u))^{1/2} f(h\gamma).$$

Then

$$\begin{aligned}
&\rho(\langle \psi \,, \varphi \rangle_\star) \\
&\quad = \int_{G(u)} \langle \psi \,, \varphi \rangle_\star (h) \rho(h) \, \mathrm{d}\beta^{G(u)}(h) \\
&\quad = \int_{G(u)} \int_G \overline{\psi(\eta)} \varphi(\eta h) \rho(h) \, \mathrm{d}\lambda_u(\eta) \, \mathrm{d}\beta^{G(u)}(h) \\
&\quad = \int_{G(u)} \int_G \overline{\psi(\eta\gamma)} \varphi(\eta\gamma h) \rho(h) \, \lambda_{r(\gamma)}(\eta) \, \mathrm{d}\beta^{G(u)}(h)
\end{aligned}$$

which, in view of Lemma 3.1, is

$$\begin{aligned}
&= \omega(\gamma, G(u)) \int_{G(r(\gamma))} \int_G \overline{\psi(\eta\gamma)} \varphi(\eta h \gamma) \gamma \cdot \rho(h) \, \mathrm{d}\lambda_{r(\gamma)}(\eta) \, \mathrm{d}\beta^{G(r(\gamma))}(h) \\
&= \gamma \cdot \rho(\langle V(\psi) \,, V(\varphi) \rangle_\star).
\end{aligned}$$

Since V is clearly onto, $f \otimes h \mapsto V(f) \otimes h$ extends to a unitary intertwining the two representations. □

Let $\check{\mathcal{K}}(P)$ be the space of the induced representation $\operatorname{ind}_{G(\sigma(P))}^G \check{r}_P$. Thus, $\check{\mathcal{K}}(P)$ is the completion of $C_c(G_{\sigma(P)}) \odot \mathcal{K}(P)$ as described above. Let

$$\operatorname{Prim} C^*(\Sigma) * \check{\mathcal{K}} = \{(P, \check{\mathcal{K}}(P)) : P \in \operatorname{Prim} C^*(\Sigma)\}$$

be the disjoint union of the $\check{\mathcal{K}}(P)$. Since

$$P \mapsto ((\varphi \otimes h)(P) \mid (\psi \otimes k)(P))$$

is Borel, there is a unique Borel structure on $\operatorname{Prim} C^*(\Sigma) * \check{\mathcal{K}}$ making it into an analytic Borel Hilbert bundle such that each $\varphi \otimes h$ defines a Borel section (see [19, Proposition F.8]). Since

$$\operatorname{Ind}_{G(\sigma(P))}^G \check{r}_P(\psi)((\varphi \otimes h)(P)) = \psi * \varphi \otimes h(P),$$

$P \mapsto \operatorname{Ind}_{G(\sigma(P))}^{G} \tilde{r}_P$ is a Borel field of representations of $C^*(G)$. Therefore, we can define the direct integral representation

$$\operatorname{ind} \tilde{r} := \int_{\operatorname{Prim} C^*(\Sigma)}^{\oplus} \operatorname{Ind}_{G(\sigma(P))}^{G} \tilde{r}_P \, d\nu(P). \tag{3.14}$$

Lemma 3.13. *Let $U \subset \operatorname{Prim} C^*(\Sigma)$ be the ν-conull set associated to the equivariant ideal center decomposition $\tilde{r}$ of r. Let $I_P := \ker(\operatorname{Ind}_{G(\sigma(P))}^{G} \tilde{r}_P)$. Then for all $P \in U$, $I_P \in \operatorname{Prim} C^*(G)$. Furthermore, $P \mapsto I_P$ is a Borel map of $U \subset \operatorname{Prim} C^*(\Sigma)$ into $\operatorname{Prim} C^*(G)$ such that $I_{P\cdot\gamma} = I_P$ for all $(P, \gamma) \in \underline{G}|_U$.*

Proof. Since $\tilde{r}_P$ has kernel P, I_P is primitive by the remarks preceding Lemma 3.12, and $P \mapsto I_P$ is Borel by [19, Lemma F.28]. Recall that by (3.9), there is a ν-conull set U such that $\tilde{r}_{P\cdot\gamma}$ is equivalent to $\gamma \cdot \tilde{r}_P$ if $(P, \gamma) \in \underline{G}|_U$. Then for $(P, \gamma) \in \underline{G}|_U$,

$$\begin{aligned} I_{P\cdot\gamma} &= \ker(\operatorname{Ind}_{G(\sigma(P\cdot\gamma))}^{G} \tilde{r}_{P\cdot\gamma}) \\ &= \ker(\operatorname{Ind}_{G(s(\gamma))}^{G} \gamma^{-1} \cdot \tilde{r}_P) \end{aligned}$$

which, by Lemma 3.12, is

$$\begin{aligned} &= \ker(\operatorname{Ind}_{G(\sigma(P))}^{G} \tilde{r}_P) \\ &= I_P. \end{aligned}$$

□

Proposition 3.14. *Let $\operatorname{ind} \tilde{r}$ be the induced representation associated to an irreducible representation R of $C^*(G)$ defined by (3.14). Then the kernel of $\operatorname{ind} \tilde{r}$ is an induced primitive ideal.*

Proof. Let $\kappa : \operatorname{Prim} C^*(\Sigma) \to \operatorname{Prim} C^*(G)$ be a Borel map such that $\kappa(P) := I_P$ for $P \in U$. If $B \subset \operatorname{Prim} C^*(G)$ is Borel, then $\kappa^{-1}(B)$ is ν-essentially invariant. Since ν is ergodic by Proposition 3.9, the proof of [19, Lemma D.47] implies that κ is essentially constant; that is, there is a P_0 such that $\ker(\operatorname{Ind}_{G(\sigma(P))}^{G} \tilde{r}_P) = I_{P_0}$ for ν-almost all P. But then $\ker(\operatorname{ind} \tilde{r}) = I_{P_0}$. This is what we wanted. □

Let r_u be as in (3.6) and let $\tilde{H}(u)$ be the space of the induced representation $\operatorname{Ind}_{G(u)}^{G} r_u$. Thus $\tilde{H}(u)$ is the completion of $C_c(G_u) \odot \mathcal{H}(u)$ with respect to the inner product

$$(\varphi \otimes h \mid \psi \otimes k) = (r_u(\langle \psi \, , \varphi \rangle_{\star})h \mid k).$$

(There is no harm in taking φ and ψ in $C_c(G)$ above.) Let $G^{(0)} * \tilde{\mathcal{H}} = \{(u, \tilde{H}(u)) : u \in G^{(0)}\}$ be the disjoint union. Then for each $\varphi \otimes h \in C_c(G) \otimes L^2(G^{(0)} * \mathcal{H}, \mu)$ we get a section of $G^{(0)} * \tilde{\mathcal{H}}$ by

$$(f \otimes h)(u) = f \otimes h(u).$$

Then

$$\begin{aligned}(3.15)\quad &(\varphi \otimes h(u) \mid \psi \otimes k(u)) \\ &\qquad = (r_u(\langle \psi \, , \, \varphi \rangle_\star) h \mid k) \\ &\qquad = \int_{G(u)} (\psi^* * \varphi(\eta) V_\eta h(u) \mid k(u)) \Delta_{G(u)}(\eta)^{-1/2} \, \mathrm{d}\beta^{G(u)}(\eta),\end{aligned}$$

which is Borel in u. Thus by [19, Proposition F.8], there is a unique Borel structure on $G^{(0)} * \tilde{\mathcal{H}}$ making it into an analytic Borel Hilbert bundle such that each $f \otimes h$ is a Borel section. Since

$$\operatorname{Ind}_{G(u)}^{G} r_u(\psi)(\varphi \otimes h(u)) = \psi * \varphi \otimes h,$$

it follows that $u \mapsto \operatorname{Ind}_{G(u)}^{G} r_u$ is a Borel field of representations and that we can make sense out of the direct integral representation

$$\operatorname{ind} r := \int_{G^{(0)}}^{\oplus} \operatorname{Ind}_{G(u)}^{G} r_u \, \mathrm{d}\mu(u)$$

on $L^2(G^{(0)} * \tilde{\mathcal{H}}, \mu)$.

Proof of Theorem 2.1. Let $\underline{L}$ be the induced representation of $C^*(G)$ constructed by Renault on pages 16–17 of [16]. After a bit of untangling and after specializing [16, Lemma 2.3] to our case, we see that there is a unitary U mapping the space of $\underline{L}$ onto the completion of $C_c(G) \odot L^2(G^{(0)} * \mathcal{H}, \mu)$ with respect to the inner product

$$(\varphi \otimes h \mid \psi \otimes k) = \int_{G^{(0)}} (\varphi \otimes h(u) \mid \psi \otimes k(u)) \, \mathrm{d}\mu(u),$$

where the integrand on the right-hand side is given by (3.15). Moreover,

$$(U^* \underline{L}(\psi) U)(\varphi \otimes h) = \psi * \varphi \otimes h.$$

Simply said: $\underline{L}$ is equivalent to $\operatorname{ind} r$.

Let $M^R : L^2(G^{(0)} * \mathcal{H}, \mu) \to L^2(\operatorname{Prim} C^*(\Sigma) * \mathcal{K}, \nu)$ be the unitary implementing the equivalence between r and $\tilde{r}$, and then define

$$W : C_c(G) \odot L^2(G^{(0)} * \mathcal{H}, \mu) \to C_c(G) \odot L^2(\operatorname{Prim} C^*(\Sigma) * \mathcal{K}, \nu))$$

by $W(\varphi \otimes h) := \varphi \otimes M^R h$. Since

$$\begin{aligned} W \circ \operatorname{ind} r(\psi)(\varphi \otimes h) &= W(\psi * \varphi \otimes M^R h) \\ &= \operatorname{ind} \tilde{r}(\psi)(f \otimes M^R h) \\ &= \operatorname{ind} \tilde{r}(\psi) \circ W(\varphi \otimes h),\end{aligned}$$

it is not hard to see that W extends to a unitary intertwining $\operatorname{ind} r$ and $\operatorname{ind} \tilde{r}$. Therefore, $\operatorname{ind} \tilde{r}$ and $\underline{L}$ have the same kernel. But then [16, Theorem 3.3] implies that $\ker R \subset \ker(\operatorname{ind} \tilde{r})$. On the other hand, if G is amenable as in the statement of the theorem, then [16, Theorem 3.6] implies that $K = \ker R = \ker(\operatorname{ind} \tilde{r})$. However, $\ker(\operatorname{ind} \tilde{r})$ is an induced primitive ideal by Proposition 3.14. This completes the proof. ◻

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