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C^*-Algebras Associated with Mauldin–Williams Graphs

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Abstract. A Mauldin–Williams graph $\mathcal{M}$ is a generalization of an iterated function system by a directed graph. Its invariant set K plays the role of the self-similar set. We associate a C^*-algebra $\mathcal{O}_{\mathcal{M}}(K)$ with a Mauldin–Williams graph $\mathcal{M}$ and the invariant set K, laying emphasis on the singular points. We assume that the underlying graph G has no sinks and no sources. If $\mathcal{M}$ satisfies the open set condition in K, and G is irreducible and is not a cyclic permutation, then the associated C^*-algebra $\mathcal{O}_{\mathcal{M}}(K)$ is simple and purely infinite. We calculate the K-groups for some examples including the inflation rule of the Penrose tilings.

1 Introduction

Self-similar sets are often constructed as the invariant set of iterated function systems [2,13,19]. Many other examples, such as the inflation rule of the Penrose tilings, show that graph directed generalizations of iterated function systems are also interesting, and these have been developed as Mauldin–Williams graphs [3, 11, 23]. Since we can regard graph directed iterated function systems as dynamical systems, we expect that fruitful connections exist between Mauldin–Williams graphs and C^*-algebras.

[14] Ionescu defined a C^*-correspondence $\mathcal{X}$ for a Mauldin–Williams graph $\mathcal{M}$ and showed that the Cuntz–Pimsner algebra $\mathcal{O}_{\mathcal{X}}$ (see [12, 29]) is isomorphic to the Cuntz–Krieger algebra $\mathcal{O}_G$ [8] associated with the underlying graph G. In particular, if the Mauldin–Williams graph has one vertex and N edges, the Cuntz–Pimsner algebra $\mathcal{O}_{\mathcal{X}}$ is isomorphic to the Cuntz algebra $\mathcal{O}_N$ [7], which recovers a result in [30]. The construction is useful because it gives many examples of different non-self-adjoint algebras which are not completely isometrically isomorphic, but have the same C^*-envelope [25] as shown by Ionescu [15].

On the other hand, Kajiwara and Watatani [17] introduced C^*-algebras associated with rational functions including singular points, *i.e.,* branched points. They developed the idea to associate C^*-algebras with self-similar sets considering singular points [16]. In this paper we associate another C^*-algebra $\mathcal{O}_{\mathcal{M}}(K)$ with a Mauldin–Williams graph $\mathcal{M}$ and its invariant set K putting emphasis on the singular points as above. We show that the associated C^*-algebra $\mathcal{O}_{\mathcal{M}}(K)$ is not isomorphic to the Cuntz–Krieger algebra $\mathcal{O}_G$ for the underlying graph G in general. This comes from the fact that the singular points cause the failure of the injectivity of the coding by the Markov shift for G. We assume that the underlying graph G has no sinks and no sources. We show that the associated C^*-algebra $\mathcal{O}_{\mathcal{M}}(K)$ is simple and purely infinite if $\mathcal{M}$ satisfies the open set condition in K, and if G is irreducible and is not a cyclic

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permutation. We calculate the K-groups for some examples including the inflation rule of the Penrose tilings. The C^*-algebras associated with tilings were first studied by Connes [6] and were also discussed by Mingo [24] and Anderson and Putnam [1], but we do not know the exact relation between their constructions and ours.

Our construction has a common flavor with several topological generalizations of graph C^*-algebras [5, 18, 21, 26, 27]. Since graph directed iterated function systems are sometimes obtained as continuous cross sections of expanding maps, C^*-algebras associated with interval maps introduced by Deaconu and Shultz [9] are closely related with our construction. From a different point of view, Bratteli and Jorgensen studied a relation between iterated function systems and representation of Cuntz algebras [4].

2 Mauldin–Williams Graphs and the Associated C^*-Correspondence

By a *Mauldin–Williams graph* we mean a system $\mathcal{M} = (G, \{T_v, \rho_v\}_{v\in E^0}, \{\phi_e\}_{e\in E^1})$, where $G = (E^0, E^1, r, s)$ is a graph with a finite set of vertices E^0, a finite set of edges E^1, a *range* map r, and a *source* map s, and where $\{T_v, \rho_v\}_{v\in E^0}$ and $\{\phi_e\}_{e\in E^1}$ are families such that:

(i) each T_v is a compact metric space with a prescribed metric ρ_v, $v \in E^0$;

(ii) for $e \in E$, ϕ_e is a continuous map from $T_{r(e)}$ to $T_{s(e)}$ such that

$$c'\rho_{r(e)}(x, y) \le \rho_{s(e)}(\phi_e(x), \phi_e(y)) \le c\rho_{r(e)}(x, y)$$

for some constants c', c satisfying $0 < c' < c < 1$ (independent of e) and all $x, y \in T_{r(e)}$.

We shall also assume that the source map s and the range map r are surjective. Thus, we assume that there are no sinks and no sources in the graph G.

In the particular case when we have only one vertex and N edges, we obtain a so-called *iterated function system* $(K, \{\phi_i\}_{i=1,\dots,N})$.

An *invariant list* associated with a Mauldin–Williams graph

$$\mathcal{M} = (G, \{T_v, \rho_v\}_{v\in E^0}, \{\phi_e\}_{e\in E^1})$$

is a family $(K_v)_{v\in E^0}$ of compact sets such that $K_v \subset T_v$ for all $v \in E^0$, and such that

$$K_v = \bigcup_{\substack{e\in E^1\\ s(e)=v}} \phi_e(K_{r(e)}).$$

Since each ϕ_e is a contraction, $\mathcal{M}$ has a unique invariant list (see [23, Theorem 1]). We set $T := \bigcup_{v\in E^0} T_v$ and $K := \bigcup_{v\in E^0} K_v$, and we call K the *invariant set* of the Mauldin–Williams graph.

In the particular case when we have one vertex v and N edges, *i.e.,* in the setting of an iterated function system, the invariant set is the unique compact subset $K := K_v$ of $T = T_v$ such that $K = \phi_1(K) \cup \cdots \cup \phi_N(K)$.

In this paper we forget about the ambient space T. That is, we consider that the Mauldin–Williams graph is $\mathcal{M} = (G, \{K_v, \rho_v\}_{v\in E^0}, \{\phi_e\}_{e\in E^1})$, with $K = \bigcup_{v\in E^0} K_v$ being the invariant set.

We say that a graph $G = (E^0, E^1, r, s)$ is *irreducible* (or *totally connected*) if for every $v_1, v_2 \in E^0$ there is a finite path w such that $s(w) = v_1$ and $r(w) = v_2$. We will assume in this paper that the graph $G = (E^0, E^1, r, s)$ is irreducible and not a cyclic permutation. That is, there exists a vertex $v^* \in E^0$ and two edges $e_1 \neq e_2$ such that $s(e_1) = s(e_2) = v^*$.

For a natural number m, we define

$$E^m := \{w = (w_1, \dots, w_m) : w_i \in E^1 \text{ and } r(w_i) = s(w_{i+1}) \text{ for all } i = 1, \dots, m-1\}.$$

An element $w \in E^m$ is called a *path of length m*. We extend s and r to E^m by $s(w) = s(w_1)$ and $r(w) = r(w_m)$ for all $w \in E^m$. We set $E^* = \bigcup_{m \geq 1} E^m$ and denote the length of a path w by $l(w)$. We also define $E^m(v) := \{w \in E^m \; : \; s(w) = v\}$ to be the set of paths of length m starting at the vertex v and $E^*(v) = \bigcup_{m \geq 1} E^m(v)$ to be the set of finite paths starting at v.

The infinite path space is

$$E^\infty = \{w = (w_n)_{n \geq 1} \; : \; r(w_n) = s(w_{n+1}) \text{ for all } n \geq 1\},$$

and the space of infinite paths starting at a vertex v is

$$E^\infty(v) = \{w \in E^\infty \; : \; s(w) = s(w_1) = v\}.$$

On $E^\infty(v)$ we define the metric $\delta_v(\alpha, \beta) = c^{l(\alpha \wedge \beta)}$ if $\alpha \neq \beta$ and 0 otherwise, where $\alpha \wedge \beta$ is the longest common prefix of α and β (see [10, Page 116]). Then $E^\infty(v)$ is a compact metric space, and, since E^∞ equals the disjoint union of the spaces $E^\infty(v)$, E^∞ is a compact metric space in a natural way. For $w \in E^m$, let $\phi_w = \phi_{w_1} \circ \cdots \circ \phi_{w_m}$ and $K_w = \phi_w(K_{r(w)})$. Then for any infinite path $\alpha = (\alpha_n)_{n \in \mathbb{N}}$, $\bigcap_{n \geq 1} K_{(\alpha_1, \dots, \alpha_n)}$ contains only one point. Therefore we can define a map $\pi : E^\infty \to K$ by $\{\pi(\alpha)\} = \bigcap_{n \geq 1} K_{(\alpha_1, \dots, \alpha_n)}$. Since $\pi(E^\infty)$ is also an invariant set, we have that $\pi(E^\infty) = K$. Thus π is a continuous onto map. Moreover, for any $y \in K_{v_0}$ and any neighborhood $U \subset K_{v_0}$ of y, there exist $n \in \mathbb{N}$ and $w \in E^n(v_0)$ such that

$$y \in \phi_w(K_{r(w)}) \subset U.$$

We say that a Mauldin–Williams graph $\mathcal{M}$ satisfies the *open set condition in K* if there exists a family of non-empty sets $(V_v)_{v \in E^0}$ such that $V_v \subset K_v$ for all $v \in E^0$, and such that

$$\bigcup_{\substack{e \in E^1 \\ s(e) = v}} \phi_e(V_{r(e)}) \subset V_v \quad \text{for all } v \in E^0,$$

$$\phi_e(V_{r(e)}) \bigcap \phi_f(V_{r(f)}) = \varnothing \quad \text{if } e \neq f.$$

Then $V := \bigcup_{v \in E^0} V_v$ is an open dense subset of K. Moreover, for $n \in \mathbb{N}$ and $w, v \in E^n$, if $w \neq v$ and $r(w) = r(v)$ then $\phi_w(V_{r(w)}) \bigcap \phi_v(V_{r(v)}) = \varnothing$.

For $e \in E^1$, we define the *cograph of* ϕ_e to be the set

$$\operatorname{cograph}(\phi_e) = \{(x, y) \in K_{s(e)} \times K_{r(e)} : x = \phi_e(y)\} \subset K \times K.$$

We shall consider the union

$$\mathcal{G} = \mathcal{G}(\{\phi_e : e \in E^1\}) := \bigcup_{e \in E^1} \operatorname{cograph}(\phi_e).$$

Consider the C^*-algebra $A = C(K)$, and let $X = C(\mathcal{G})$. Then X is a C^*-correspondence over A with the structure defined by the formulae

$$(a \cdot \xi \cdot b)(x, y) = a(x)\xi(x, y)b(y),$$
$$\langle \xi, \eta \rangle_A(y) = \sum_{\substack{e \in E^1 \\ y \in K_{r(e)}}} \overline{\xi(\phi_e(y), y)}\eta(\phi_e(y), y),$$

for all $a, b \in A$, $\xi, \eta \in X$, $(x, y) \in \mathcal{G}$ and $y \in K$. It is clear that the A-valued inner product is well defined. The left multiplication is given by the $*$-homomorphism $\Phi \colon A \to \mathcal{L}(X)$ such that $(\Phi(a)\xi)(x, y) = a(x)\xi(x, y)$ for $a \in A$ and $\xi \in X$. Put $\|\xi\|_2 = \|\langle \xi, \xi \rangle_A\|_\infty^{1/2}$.

For any natural number n, we define $\mathcal{G}_n = \mathcal{G}(\{\phi_w : w \in E^n\})$ and a C^*-correspondence $X_n = C(\mathcal{G}_n)$ similarly. We also define a path space $\mathcal{P}_n$ of length n by

$$\mathcal{P}_n = \big\{ (\phi_{w_1,\dots,w_n}(y), \phi_{w_2,\dots,w_n}(y), \dots, \phi_{w_n}(y), y) \in K^{n+1} : \\ w = (w_1, \dots, w_n) \in E^n, y \in K_{r(w)} \big\}.$$

Then $Y_n := C(\mathcal{P}_n)$ is a C^*-correspondence over A with an A-valued inner product defined by

$$\langle \xi, \eta \rangle_A(y) = \sum_{\substack{w \in E^n \\ y \in K_{r(w)}}} \overline{\xi(\phi_{w_1,\dots,w_n}(y), \dots, \phi_{w_n}(y), y)}\eta(\phi_{w_1,\dots,w_n}(y), \dots, \phi_{w_n}(y), y),$$

for all $\xi, \eta \in Y_n$ and $y \in K$.

Proposition 2.1 *Let $\mathcal{M} = (G, \{K_v, \rho_v\}_{v \in E^0}, \{\phi_e\}_{e \in E^1})$ be a Mauldin–Williams graph. Let K be the invariant set. Then $X = C(\mathcal{G})$ is a full C^*-correspondence over $A = C(K)$ without completion. The left action $\Phi \colon A \to \mathcal{L}(X)$ is unital and faithful. Similar statements hold for $Y_n = C(\mathcal{P}_n)$.*

Proof For any $\xi \in X$ we have

$$\|\xi\|_\infty \le \|\xi\|_2 = \Bigg(\sup_{y \in K} \sum_{\substack{e \in E^1 \\ y \in K_{r(e)}}} |\xi(\phi_e(y), y)|^2 \Bigg)^{1/2} \le \sqrt{N}\|\xi\|_\infty,$$

where N is the number of edges in E^1. Therefore the norms $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are equivalent. Since $C(\mathcal{G})$ is complete with respect to $\|\cdot\|_\infty$, it is also complete with respect to $\|\cdot\|_2$.

Let $\xi \in X$ be defined by the formula

$$\xi(x,y) = \frac{1}{\sqrt{\#(e : y \in K_{r(e)})}} \quad \text{for all } (x,y) \in \mathcal{G}.$$

Then $\langle \xi, \xi \rangle_A(y) = 1$, hence $\langle X, X \rangle_A$ contains the identity of A. Therefore X is full. If $a \in A$ is not zero, then there exists $x_0 \in K$ such that $a(x_0) \neq 0$. Since K is the invariant set of the Mauldin–Williams graph, there exists $e \in E^1$ and $y_0 \in K_{r(e)}$ such that $x_0 \in K_{s(e)}$ and $\phi_e(y_0) = x_0$. Choose $\xi \in X$ such that $\xi(x_0, y_0) \neq 0$. Then $\Phi(a)\xi \neq 0$, hence Φ is faithful. The statements for Y_n are similarly proved. ∎

Definition 2.2 Let $\mathcal{M} = (G, \{K_v, \rho_v\}_{v \in E^0}, \{\phi_e\}_{e \in E^1})$ be a Mauldin–Williams graph with the invariant set K. We associate a C^*-algebra $\mathcal{O}_{\mathcal{M}}(K)$ to $\mathcal{M}$ as the Cuntz–Pimsner algebra $\mathcal{O}_X$ of the C^*-correspondence $X = C(\mathcal{G})$ over the C^*-algebra $A = C(K)$.

As in [16], we denote by $\mathcal{O}_X^{op}$ the $*$-algebra generated algebraically by A and S_ξ with $\xi \in X$. The *gauge action* is $\gamma \colon \mathbb{R} \to \operatorname{Aut} \mathcal{O}_X$ defined by $\gamma_t(S_\xi) = e^{it} S_\xi$ for all $\xi \in X$, and $\gamma_t(a) = a$ for all $a \in A$.

Proposition 2.3 *Let $\mathcal{M} = (G, \{K_v, \rho_v\}_{v \in E^0}, \{\phi_e\}_{e \in E^1})$ be a Mauldin–Williams graph. Assume that K is the invariant set of the graph. Then there is an isomorphism $\varphi_n \colon X^{\otimes n} \to C(\mathcal{P}_n)$, as C^*-correspondences over A, such that*

$$\begin{aligned}(\varphi_n(\xi_1 \otimes \cdots \otimes \xi_n))&(\phi_{w_1,\ldots,w_n}(y), \ldots, \phi_{w_n}(y), y) \\ &= \xi_1(\phi_{w_1,\ldots,w_n}(y), \phi_{w_2,\ldots,w_n}(y)) \xi_2(\phi_{w_2,\ldots,w_n}(y), \phi_{w_3,\ldots,w_n}(y)) \cdots \xi_n(\phi_{w_n}(y), y)\end{aligned}$$

for all $\xi_1, \ldots, \xi_n \in X$, $y \in K$, and $w = (w_1, \ldots, w_n) \in E^n$ such that $y \in K_{r(w_n)}$. Moreover, let $\rho_n \colon \mathcal{P}_n \to \mathcal{G}_n$ be an onto continuous map such that

$$\rho_n(\phi_{w_1,\ldots,w_n}(y), \ldots, \phi_{w_n}(y), y) = (\phi_{w_1,\ldots,w_n}(y), y).$$

Then $\rho_n^* : C(\mathcal{G}_n) \ni f \to f \circ \rho_n \in C(\mathcal{P}_n)$ is an embedding as a Hilbert submodule preserving inner product.

Proof It is easy to see that φ_n is well-defined and a bimodule morphism. We show that φ_n preserves the inner product. Consider the case when $n = 2$ for simplicity of notation. Let $\xi_1, \xi_2, \eta_1, \eta_2 \in X$. We have

$$\begin{aligned}\langle \xi_1 \otimes \xi_2, &\eta_1 \otimes \eta_2 \rangle_A(y) \\ &= \langle \xi_2, \langle \xi_1, \eta_1 \rangle_A \cdot \eta_2 \rangle_A(y) \\ &= \sum_{\substack{e \in E \\ y \in K_{r(e)}}} \overline{\xi_2(\phi_e(y), y)} \langle \xi_1, \eta_1 \rangle_A(\phi_e(y)) \eta_2(\phi_e(y), y) \\ &= \sum_{\substack{fe \in E^2 \\ y \in K_{r(e)}}} \overline{\xi_1(\phi_{fe}(y), \phi_e(y)) \xi_2(\phi_e(y), y)} \eta_1(\phi_{fe}(y), \phi_e(y)) \eta_2(\phi_e(y), y) \\ &= \langle \varphi_2(\xi_1 \otimes \xi_2), \varphi_2(\eta_1 \otimes \eta_2) \rangle_A(y).\end{aligned}$$

Since φ_n preserves the inner product, it is one to one. Using the Stone-Weierstrass Theorem, one can show that φ_n is also onto. The rest is clear. ■

We let $i_{n,m}\colon C(\mathcal{P}_n) \to C(\mathcal{P}_m)$ be the natural inner-product preserving embedding, for $m \geq n$.

Definition 2.4 Consider a covering map $\pi\colon \mathcal{G} \to K$ defined by $\pi(x, y) = y$ for $(x, y) \in \mathcal{G}$. Define the set

$$B(\mathcal{M}) := \{x \in K : x = \phi_e(y) = \phi_f(y) \text{ for some } y \in K \text{ and } e \neq f\}.$$

The set $B(\mathcal{M})$ will be described by the ideal $I_X := \Phi^{-1}(\mathcal{K}(X))$ of A. We define a branch index $e(x, y)$ at $(x, y) \in \mathcal{G}$ by

$$e(x, y) := \#\{e \in E^1 : \phi_e(y) = x\}.$$

Hence $x \in B(\mathcal{M})$ if and only if there exists some $y \in K$ with $e(x, y) \geq 2$. For $x \in K$ we define

$$I(x) := \{e \in E^1 : \text{there exists } y \in K \text{ such that } x = \phi_e(y)\}.$$

Lemma 2.5 *In the above situation, if $x \in K \setminus B(\mathcal{M})$, then there exists an open neighborhood U_x of x satisfying the following.*

(i) $U_x \bigcap B(\mathcal{M}) = \varnothing$.
(ii) *If $e \in I(x)$, then $\phi_f(\phi_e^{-1}(U_x)) \bigcap U_x = \varnothing$ for $e \neq f$, such that $r(e) = r(f)$.*
(iii) *If $e \notin I(x)$, then $U_x \bigcap \phi_e(K_{r(e)}) = \varnothing$.*

Proof Let $x \in K \setminus B(\mathcal{M})$. Let $v_0 \in E^0$ such that $x \in K_{v_0}$. Since $B(\mathcal{M})$ and $\bigcup_{e \notin I(x)} \phi_e(K_{r(e)})$ are closed, and x is not in either of them, there exists an open neighbourhood $W_x \subset K_{v_0}$ of x such that

$$W_x \bigcap (B(\mathcal{M}) \cup \bigcup_{e \notin I(x)} \phi_e(K_{r(e)})) = \varnothing.$$

For each $e \in I(x)$ there exists a unique $y_e \in K$ with $x = \phi_e(y_e)$, since $x \notin B(\mathcal{M})$. For $f \in E^1$, if $r(e) = r(f)$ and $f \neq e$ then $\phi_f(y_e) \neq \phi_e(y_e) = x$. Therefore there exists an open neighborhood V_x^e of x such that $\phi_f(\phi_e^{-1}(V_x^e)) \bigcap V_x^e = \varnothing$ if $f \neq e$ and $r(f) = r(e)$. Let $U_x := W_x \bigcap (\bigcap_{e \in I(x)} V_x^e)$. Then U_x is an open neighborhood of x and satisfies all the requirements. ■

Proposition 2.6 *Let $\mathcal{M} = (G, \{K_v, \rho_v\}_{v \in E^0}, \{\phi_e\}_{e \in E^1})$ be a Mauldin–Williams graph. Assume that the system $\mathcal{M}$ satisfies the open set condition in K. Then*

$$I_X = \{a \in A = C(K) : a \text{ vanishes on } B(\mathcal{M})\}.$$

Proof The proof requires only minor modifications from the proof of [16, Proposition 2.4]. ■

Corollary 2.7 $\#B(\mathcal{M}) = \dim(A/I_X)$.

Corollary 2.8 *The closed set $B(\mathcal{M})$ is empty if and only if $\Phi(A)$ is contained in $\mathcal{K}(X)$ if and only if X is a finitely generated projective right A module.*

3 Simplicity and Pure Infinitness

Let $\mathcal{M} = (G, \{K_v, \rho_v\}_{v\in E^0}, \{\phi_e\}_{e\in E^1})$ be a Mauldin–Williams graph. Let $A = C(K)$ and $X = C(\mathcal{G})$. For $e \in E^1$ define an endomorphism $\beta_e : A \to A$ by

$$(\beta_e(a))(y) := \begin{cases} a(\phi_e(y)) & \text{if } y \in K_{r(e)}, \\ 0 & \text{otherwise,} \end{cases}$$

for all $a \in A$ and $y \in K$. We also define a unital completely positive map $E_{\mathcal{M}} : A \to A$ by

$$\begin{aligned}(E_{\mathcal{M}}(a))(y) &:= \frac{1}{\#\{e \in E^1 : y \in K_{r(e)}\}} \sum_{\substack{e\in E^1 \\ y\in K_{r(e)}}} a(\phi_e(y)) \\ &= \frac{1}{\#\{e \in E^1 : y \in K_{r(e)}\}} \sum_{\substack{e\in E^1 \\ y\in K_{r(e)}}} \beta_e(a)(y),\end{aligned}$$

for $a \in A$, $y \in K$. For the function $\xi_0 \in X$ defined by the formula

$$\xi_0(x, y) = \frac{1}{\sqrt{\#\{e \in E^1 : y \in K_{r(e)}\}}},$$

we have $E_{\mathcal{M}}(a) = \langle \xi_0, \Phi(a)\xi_0\rangle_A$ and $E_{\mathcal{M}}(I) = \langle \xi_0, \xi_0\rangle_A = I$. Let $D := S_{\xi_0} \in \mathcal{O}_{\mathcal{M}}(K)$.

Lemma 3.1 *In the above situation, for $a \in A$, we have that $D^*aD = E_{\mathcal{M}}(a)$ and in particular $D^*D = I$.*

Proof The same as [16, Lemma 3.1]. ■

Definition 3.2 Let $\mathcal{M} = (G, \{K_v, \rho_v\}_{v\in E^0}, \{\phi_e\}_{e\in E^1})$ be a Mauldin–Williams graph. We say that an element $a \in A = C(K)$ is $(\mathcal{M}, E^n)$-invariant if

$$a(\phi_\alpha(y)) = a(\phi_\beta(y)) \text{ for any } y \in K \text{ and } \alpha, \beta \in E^n \text{ such that } y \in K_{r(\alpha)} = K_{r(\beta)}.$$

If $a \in A$ is $(\mathcal{M}, E^n)$-invariant, then a is also $(\mathcal{M}, E^{n-1})$-invariant. Then, as [16, Definition, p. 11], if a is $(\mathcal{M}, E^n)$-invariant, we can define

$$\beta^k(a)(y) := a(\phi_{w_1} \cdots \phi_{w_n}(y)), \qquad \text{for any } w \in E^k, \text{ such that } y \in K_{r(w_n)}.$$

Then for any $\xi_1, \dots, \xi_n \in X$ and $a \in A$ $(\mathcal{M}, E^n)$-invariant , we have the relation:

$$aS_{\xi_1} \cdots S_{\xi_n} = S_{\xi_1} \cdots S_{\xi_n}\beta^n(a).$$

Lemma 3.3 *Let $\mathcal{M} = (G, \{K_v, \rho_v\}_{v\in E^0}, \{\phi_e\}_{e\in E^1})$ be a Mauldin–Williams graph such that (E^0, E^1, r, s) is an irreducible graph. For any non-zero positive element $a \in A$ and for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ and $\xi \in X^{\otimes n}$ with $\langle \xi, \xi\rangle_A = 1$ such that*

$$\|a\| - \varepsilon \le S_\xi^* a S_\xi \le \|a\|.$$

Proof Let $x_0 \in K$ be such that $\|a\| = a(x_0)$. Let $v_0 \in E^0$ such that $x_0 \in K_{v_0}$. Then there exists an open neighborhood U_0 of x_0 in K_{v_0} such that for any $x \in U_0$ we have $\|a\| - \varepsilon \leq a(x) \leq \|a\|$. Let U_1 be an open neighborhood of x_0 in K_{v_0}, and K_1 compact such that $U_1 \subset K_1 \subset U_0$. Since the map $\pi\colon E^\infty \to K$ is onto and continuous, there exists some $n_1 \in \mathbb{N}$ and $\alpha \in E^{n_1}(v_0)$ such that $\phi_\alpha(K_{r(\alpha)}) \subset U_1$. For any vertex $v \in V$, since the graph G is irreducible, there exists a path $w_v \in E^*$ from $r(\alpha)$ to v. Then $\phi_{\alpha w_v}(K_v) \subset U_1$. Hence, for each $v \in V$, there is $\alpha_v \in E^{n(v)}(v_0)$, for some $n(v) \geq n_1$, such that $\phi_{\alpha_v}(K_v) \subset U_1$. For each $v \in V$, define the closed subsets $F_{1,v}$ and $F_{2,v}$ of $K \times K$ by

$$F_{1,v} = \{(x, y) \in K \times K : x = \phi_\alpha(y), x \in K_1, y \in K_v, \alpha \in E^{n(v)}(v_0)\},$$

$$F_{2,v} = \{(x, y) \in K \times K : x = \phi_\alpha(y), x \in U_0^c, y \in K_v, \alpha \in E^{n(v)}(v_0)\}.$$

Since $F_{1,v} \cap F_{2,w} = \varnothing$ for all $w \in V$ and $F_{1,v} \cap F_{1,w} = \varnothing$ if $v \neq w$, there exists $g_v \in C(\mathcal{G}_{n(v)})$ such that $0 \leq g_v(x, y) \leq 1$ and

$$g_v(x, y) = \begin{cases} 1 & \text{if } (x, y) \in F_{1,v}, \\ 0 & \text{if } (x, y) \in \bigcup_{w \in E^0} F_{2,w} \cup \bigcup_{w \neq v} F_{1,w}. \end{cases}$$

Since $\phi_{\alpha_v}(K_v) \subset U_1$ for each $y \in K_v$, there exists $x_y \in U_1$ such that $x_y = \phi_{\alpha_v}(y) \in U_1 \subset K_1$. Therefore

$$\langle g_v, g_v\rangle_A(y) = \sum_{\substack{\alpha \in E^{n(v)} \\ y \in K_{r(\alpha)}}} |g_v(\phi_\alpha(y), y)|^2 \geq |g_v(x_y, y)|^2 = 1$$

for all $y \in K_v$. Let $n = \max\{n(v) : v \in E^0\}$. We identify $X^{\otimes n}$ with $C(\mathcal{P}_n)$ as in Proposition 2.3. We denote $i_{n(v),n}(\rho^*_{n(v)}(g_v)) \in C(\mathcal{P}_n)$ also by g_v for each $v \in V$. Let $g := \sum_{v \in V} g_v \in C(\mathcal{P}_n)$. Then $\langle g, g\rangle_A(y) \geq 1$ for all $y \in K$. Thus $b := \langle g, g\rangle_A$ is positive and invertible. Let $\xi := gb^{-1/2} \in X^{\otimes n}$. Then

$$\langle \xi, \xi\rangle_A = \langle gb^{-1/2}, gb^{-1/2}\rangle_A = b^{-1/2}\langle g, g\rangle_A b^{-1/2} = 1_A.$$

For any $y \in K$ and any $\alpha \in E^n$ such that $y \in K_{r(\alpha)}$, let $x = \phi_\alpha(y)$. If $x \in U_0$, then $\|a\| - \varepsilon \leq a(x)$, and, if $x \in U_0^c$, then

$$\xi(\phi_{\alpha_1,\ldots,\alpha_n}(y), \ldots, \phi_{\alpha_n}(y), y) = g(x, y)b^{-1/2}(y) = 0.$$

Therefore

$$\begin{aligned} \|a\| - \varepsilon &= (\|a\| - \varepsilon)\langle \xi, \xi\rangle_A(y) \\ &= (\|a\| - \varepsilon) \sum_{\substack{\alpha \in E^n \\ y \in K_{r(\alpha)}}} |\xi(\phi_{\alpha_1,\ldots,\alpha_n}(y), \ldots, \phi_{\alpha_n}(y), y)|^2 \\ &\leq \sum_{\substack{\alpha \in E^n \\ y \in K_{r(\alpha)}}} a(\phi_\alpha(y))|\xi(\phi_{\alpha_1,\ldots,\alpha_n}(y), \ldots, \phi_{\alpha_n}(y), y)|^2 \\ &= \langle \xi, a\xi\rangle_A(y) = S_\xi^* a S_\xi(y). \end{aligned}$$

We also have that $S_\xi^* a S_\xi = \langle \xi, a\xi\rangle_A \leq \|a\|\langle \xi, \xi\rangle_A = \|a\|$. ∎

Lemma 3.4 *Let $\mathcal{M} = (G, \{K_v, \rho_v\}_{v\in E^0}, \{\phi_e\}_{e\in E^1})$ be a Mauldin–Williams graph. Assume that K is the invariant set of the graph. For any non-zero positive element $a \in A$ and for any $\varepsilon > 0$ with $0 < \varepsilon < \|a\|$, there exists $n \in \mathbb{N}$ and $u \in X^{\otimes n}$ such that*

$$\|u\|_2 \le (\|a\| - \varepsilon)^{-1/2} \text{ and } S_u^* a S_u = 1.$$

Proof The proof is identical with the proof of [16, Lemma 3.4]. ■

Lemma 3.5 *Let $\mathcal{M} = (G, \{K_v, \rho_v\}_{v\in E^0}, \{\phi_e\}_{e\in E^1})$ be a Mauldin–Williams graph. Suppose that the graph G has no sinks and no sources, and it is irreducible and not a cyclic permutation. Assume that K is the invariant set of the graph and that $\mathcal{M}$ satisfies the open set condition in K. For any $n \in \mathbb{N}$, for any $T \in \mathcal{L}(X^{\otimes n})$, and for any $\varepsilon > 0$ there exists a positive element $a \in A$ such that a is $(\mathcal{M}, E^n)$-invariant,*

$$\|\Phi(a)T\|^2 \ge \|T\|^2 - \varepsilon,$$

and $\beta^p(a)\beta^q(a) = 0$ for $p, q = 1, \ldots, n$ with $p \ne q$.

Proof Let $n \in \mathbb{N}$, let $T \in \mathcal{L}(X^{\otimes n})$, and let $\varepsilon > 0$. Then there exists $\xi \in X^{\otimes n}$ such that $\|\xi\|_2 = 1$ and $\|T\|^2 \ge \|T\xi\|_2^2 > \|T\|^2 - \varepsilon$. Hence there exists $y_0 \in K_{v_0}$ for some $v_0 \in V$ such that

$$\|T\xi\|_2^2 = \sum_{\substack{\alpha\in E^n \\ r(\alpha)=v_0}} |(T\xi)(\phi_{\alpha_1,\ldots,\alpha_n}(y_0), \ldots, \phi_{\alpha_n}(y_0), y_0)|^2 > \|T\|^2 - \varepsilon.$$

Then there exists an open neighborhood U_0 of y_0 in K_{v_0} such that for any $y \in U_0$

$$\sum_{\substack{\alpha\in E^n \\ r(\alpha)=v_0}} |(T\xi)(\phi_{\alpha_1,\ldots,\alpha_n}(y), \ldots, \phi_{\alpha_n}(y), y)|^2 > \|T\|^2 - \varepsilon.$$

Since $\mathcal{M}$ satisfies the open set condition in K, there exists a family of non-empty sets $(V_v)_{v\in E^0}$, such that $V_v \subset K_v$, for all $v \in E^0$,

$$\bigcup_{\substack{e\in E^1 \\ s(e)=v}} \phi_e(V_{r(e)}) \subset V_v \text{ for all } v \in E^0,$$

$$\phi_e(V_{r(e)}) \cap \phi_f(V_{r(f)}) = \varnothing \text{ if } e \ne f.$$

Then there exists $y_1 \in V_{v_0} \cap U_0$ and an open neighborhood U_1 of y_1 with $U_1 \subset V \cap U_0$. Moreover, there is some $k' \in \mathbb{N}$ and $(e_1, \ldots, e_{k'}) \in E^{k'}(v_0)$ such that

$$\phi_{e_1,\ldots,e_{k'}}(V_{r(e_{k'})}) \subset U_1 \subset V_{v_0} \cap U_0.$$

Since the graph G is not a cyclic permutation, there is a vertex $v^* \in E^0$ and two edges $e', e'' \in E^1$ such that $e' \ne e''$. Since the graph G is irreducible, there exists a path from $r(e_{k'})$ to v^*. Hence we have a path $(e_1, \ldots, e_k) \in E^k(v_0)$ for some $k \in \mathbb{N}$, $k \ge k'$, such that $r(e_k) = v^*$ and $\phi_{e_1,\dots,e_k}(V_{r(e_k)}) \subset U_1 \subset V_{v_0} \cap U_0$. Then we can find a path $(e_{k+1}, \dots, e_{k+n}) \in E^n(v^*)$ such that $e_{k+1} \neq e_{k+i}$ if $i \neq 1$. To see this, let $e_{k+1} = e'$. If $r(e_{k+1}) = v^*$, take $e_{k+2} = e''$. If $r(e_{k+1}) \neq v^*$, since G has no sinks, there is an edge $e \in E^1$ such that $s(e) = r(e_{k+1})$. Then $e \neq e_{k+1}$. Let $e_{k+2} = e$. If $r(e_{k+2}) = v^*$, take $e_{k+3} = e''$; if $r(e_{k+2}) \neq v^*$, take e_{k+3} to be any edge such that $s(e_{k+3}) = r(e_{k+2})$. Therefore $e_{k+3} \neq e_{k+1}$. Inductively, we obtain the path $(e_{k+1}, \dots, e_{k+n}) \in E^n(v^*)$ with the desired property. Then

$$\varnothing \neq \phi_{e_1,\dots,e_{k+n}}(V_{r(e_{k+n})}) \subset U_1 \subset V_{v_0} \cap U_0.$$

There exist $y_2 \in U_1$, an open neighborhood U_2 of y_2 in K_{v_0} and a compact set L such that

$$y_2 \in U_2 \subset L \subset \phi_{e_1,\dots,e_{k+n}}(V_{r(e_{k+n})}) \subset U_1 \subset V_{v_0} \cap U_0.$$

Let $b \in A$ such that $0 \leq b \leq 1$, $b(y_2) = 1$ and $b|_{U_2^c} = 0$. For $\alpha \in E^n$ such that $r(\alpha) = v_0$, we have

$$\phi_\alpha(y_2) \in \phi_\alpha(U_2) \subseteq \phi_\alpha(L) \subseteq \phi_\alpha(V_{v_0}).$$

Moreover, for $\alpha, \beta \in E^n$ such that $r(\alpha) = r(\beta) = v_0$, by the open set condition,

$$\phi_\alpha(L) \cap \phi_\beta(L) = \varnothing \text{ if } \alpha \neq \beta.$$

We define a positive function a on K by the formula

$$a(x) = \begin{cases} b(\phi_\alpha^{-1}(x)) & \text{if } x \in \phi_\alpha(L),\ \alpha \in E^n \text{ such that } r(\alpha) = v_0, \\ 0 & \text{otherwise.} \end{cases}$$

Then a is continuous on K, hence $a \in A$. By construction, a is $(\mathcal{M}, E^n)$-invariant.

Let $p \leq n$ be a natural number. Let $(\alpha_1, \dots, \alpha_p) \in E^p$. If there is no path $(\alpha_{p+1}, \dots, \alpha_n) \in E^{n-p}(r(\alpha_p))$ such that $r(\alpha_n) = v_0$, then $\beta_{\alpha_p} \dots \beta_{\alpha_1}(a) = 0$. If there is at least one path $(\alpha_{p+1}, \dots, \alpha_n) \in E^{n-p}(r(\alpha_p))$ such that $r(\alpha_n) = v_0$, then

$$\operatorname{supp}(\beta_{\alpha_p} \cdots \beta_{\alpha_1}(a)) \subseteq \bigcup_{\substack{(\alpha_{p+1},\cdots,\alpha_n) \in E^{n-p}(r(\alpha_p)) \\ r(\alpha_n) = v_0}} \phi_{\alpha_{p+1}\dots\alpha_n}(\operatorname{supp}(b)).$$

Since $\operatorname{supp}(b) \subseteq L \subset \phi_{e_1\dots e_{k+n}}(V_{r(e_{k+n})})$ we have that

$$\operatorname{supp}(\beta_{\alpha_p} \cdots \beta_{\alpha_1}(a)) \subseteq \bigcup_{\substack{(\alpha_{p+1},\dots,\alpha_n) \in E^{n-p}(r(\alpha_p)) \\ r(\alpha_n) = v_0}} \phi_{\alpha_{p+1}\cdots\alpha_n}\phi_{e_1\cdots e_{k+n}}(V_{r(e_{k+n})}).$$

Then, for $1 \leq p < q \leq n$ we have that

$$\operatorname{supp}(\beta^p(a)) \subseteq \bigcup_{\substack{(\alpha_{p+1},\cdots,\alpha_n) \in E^{n-p} \\ r(\alpha_n) = v_0}} \phi_{\alpha_{p+1}\cdots\alpha_n}\phi_{e_1\cdots e_{k+n}}(V_{r(e_{k+n})})$$

and

$$\operatorname{supp}(\beta^q(a)) \subseteq \bigcup_{\substack{(\alpha_{q+1},\dots,\alpha_n)\in E^{n-q}\\ r(\alpha_n)=v_0}} \phi_{\alpha_{q+1}\cdots\alpha_n}\phi_{e_1\cdots e_{k+n}}(V_{r(e_{k+n})}).$$

Since the $(n-p)+(k+1)$-th subsuffixes are different, as $e_{k+1} \neq e_{k+1+(q-p)}$, we have that $\operatorname{supp}(\beta^p(a)) \cap \operatorname{supp}(\beta^q(a)) = \varnothing$. Hence $\beta^p(a)\beta^q(a) = 0$.

Moreover, we have

$$\begin{aligned}
\|\Phi(a)T\xi\|_2^2 &= \sup_{y\in K} \sum_{\substack{\alpha\in E^n\\ y\in K_{r(\alpha)}}} |\ (a(\phi_\alpha(y))(T\xi)(\phi_\alpha(y),\dots,\phi_{\alpha_n}(y),y)\ |^2\\
&= \sup_{y\in L} \sum_{\substack{\alpha\in E^n\\ y\in K_{r(\alpha)}}} |\ (b(y))(T\xi)(\phi_\alpha(y),\dots,\phi_{\alpha_n}(y),y)\ |^2\\
&\geq \sum_{\substack{\alpha\in E^n\\ y_2\in K_{r(\alpha)}}} |\ (T\xi)(\phi_\alpha(y_2),\dots,\phi_{\alpha_n}(y_2),y_2)\ |^2\\
&> \|T\|^2 - \varepsilon,
\end{aligned}$$

because $y_2 \in U_0$. Thus $\|\Phi(a)T\|^2 \geq \|T\|^2 - \varepsilon$. ■

As in [16], we let $\mathcal{F}_n$ be the C^*-subalgebra of $\mathcal{F}_X$ generated by $\mathcal{K}(X^{\otimes k})$, $k = 0, 1, \dots, n$, and we let B_n be the C^*-subalgebra of $\mathcal{O}_X$ generated by

$$\bigcup_{i=1}^{n} \{S_{x_1}\cdots S_{x_k}S_{y_k}^*\cdots S_{y_1}^* : x_1,\dots,x_k,y_1,\dots,y_k \in X\} \cup A.$$

We will also use the isomorphism $\varphi : \mathcal{F}_n \to B_n$ such that

$$\varphi(\theta_{x_1\otimes\cdots\otimes x_k, y_1\otimes\cdots\otimes y_k}) = S_{x_1}\cdots S_{x_k}S_{y_k}^*\cdots S_{y_1}^*.$$

Lemma 3.6 *In the above situation, let $b = c^*c$ for some $c \in \mathcal{O}_X^{alg}$. We decompose $b = \sum_j b_j$ with $\gamma_t(b_j) = e^{ijt}b_j$. For any $\varepsilon > 0$ there exists $P \in A$ with $0 \leq P \leq I$ satisfying the following:*

(i) $Pb_jP = 0$ *($j \neq 0$);*
(ii) $\|Pb_0P\| \geq \|b_0\| - \varepsilon$.

Proof The proof requires only small modifications from the proof of [16, Lemma 3.6]. ■

Having proved the equivalent of the [16, Lemma 3.1–Lemma 3.6], we obtain, using the same proof as [16, Theorem 3.7], the corresponding result for the Mauldin–Williams graph.

Theorem 3.7 *Let $\mathcal{M} = (G, \{K_v, \rho_v\}_{v\in E^0}, \{\phi_e\}_{e\in E^1})$ be a Mauldin–Williams graph. Suppose that the graph G has no sinks and no sources, is irreducible, and is not a cyclic permutation. Assume that K is the invariant set of the Mauldin–Williams graph and that $\mathcal{M}$ satisfies the open set condition in K. Then the associated C^*-algebra $\mathcal{O}_{\mathcal{M}}(K)$ is simple and purely infinite.*

Using the same argument as in [16, Proposition 3.8], one can show that $\mathcal{O}_{\mathcal{M}}(K)$ is separable and nuclear and satisfies the universal coefficient theorem. Thus, by the classification theorems of Kirchberg and Phillips [20, 28], the isomorphism class of $\mathcal{O}_{\mathcal{M}}(K)$ is completely determined by the K-theory with the class of the unit.

4 Examples

We will compute the K-groups of the C^*-algebra associated with a graph $G = (E^0, E^1, r, s)$ using the fact that $K_1(C^*(G))$ is isomorphic to the kernel of $1 - A_G^t\colon \mathbb{Z}^{E^0} \to \mathbb{Z}^{E^0}$, and $K_0(C^*(G))$ is isomorphic to the cokernel of the same map, where A_G is the vertex $E^0 \times E^0$ matrix defined by

$$A_G(v, w) = \#\{e \in E^1 : s(e) = v \text{ and } r(e) = w\}.$$

For the K-groups of the Cuntz–Pimsner algebras we will use the following six-term exact sequence due to Pimsner [29] (see also [16, §4]):

$$\begin{tikzcd}
K_0(I_X) \arrow[r, "\mathrm{id} - [X]"] & K_0 A \arrow[r, "i_*"] & K_0(\mathcal{O}_M(K)) \arrow[d, "\delta_0"] \\
K_1(\mathcal{O}_M(K)) \arrow[u, "\delta_1"] & K_1(A) \arrow[l, "i_*"] & K_1(I_X) \arrow[l, "\mathrm{id} - [X]"]
\end{tikzcd}$$

First we give a condition, which is similar to the one in [16, §4], that implies that the associated C^*-algebra $\mathcal{O}_{\mathcal{M}}(K)$ is isomorphic to the C^*-algebra associated with the underlying graph.

Definition 4.1 We say that a Mauldin–Williams graph

$$\mathcal{M} = (G, \{K_v, \rho_v\}_{v \in E^0}, \{\phi_e\}_{e \in E^1})$$

satisfies the *graph separation condition* in K if K is the invariant set of the Mauldin–Williams graph and $\mathrm{cograph}(\phi_e) \cap \mathrm{cograph}(\phi_f) = \varnothing$ if $e \neq f$.

Proposition 4.2 *Let $\mathcal{M} = (G, \{K_v, \rho_v\}_{v \in E^0}, \{\phi_e\}_{e \in E^1})$ be a Mauldin–Williams graph which satisfies the graph separation condition. Then the associated C^*-algebra is isomorphic to the C^*-algebra associated with the underlying graph G.*

Proof Let $E^1 \times_G K = \{(e, x) \mid x \in K_{r(e)}\}$. Let $\mathcal{X}$ be the C^*-correspondence over $A = C(K)$ associated in [14] with a Mauldin–Williams graph. That is, $\mathcal{X} = C(E^1 \times_G K)$ with the operations given by the formulae:

$$\xi \cdot a(e, x) := \xi(e, x) a(x), \quad a \cdot \xi(e, x) := a \circ \phi_e(x) \xi(e, x),$$

where $a \in C(K)$ and $\xi \in \mathcal{X}$, and

$$\langle \xi, \eta \rangle_A(x) := \sum_{\substack{e \in E^1 \\ x \in K_{r(e)}}} \overline{\xi(e, x)} \eta(e, x),$$

for $\xi, \eta \in \mathcal{X}$. It was proved in [14, Theorem 2.3] that the Cuntz–Pimsner algebra associated with this C^*-correspondence is isomorphic to the Cuntz–Krieger C^*-algebra $C^*(G)$ of the underlying graph G. We will show that if the Mauldin–Williams graph satisfies the graph separation condition, then X and $\mathcal{X}$ are isomorphic as C^*-correspondences in the sense of [26].

Let $V : X \to \mathcal{X}$ defined by the formula $(V\xi)(e,x) = \xi(\phi_e(x),x)$ for all $(e,x) \in E^1 \times_G K$. Then

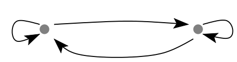
$$\begin{aligned}(V(a\cdot\xi\cdot b))(e,x) &= (a\cdot\xi\cdot b)(\phi_e(x),x)\\ &= a(\phi_e(x))\xi(\phi_e(x),x)b(x) = (a\cdot V(\xi)\cdot b)(e,x),\end{aligned}$$

for all $a, b \in A$ and $\xi \in X$. Also

$$\begin{aligned}\langle V(\xi), V(\eta)\rangle_A(x) &= \sum_{\substack{e\in E^1\\ x\in K_{r(e)}}} \overline{V(\xi)(e,x)}V(\eta)(e,x)\\ &= \sum_{\substack{e\in E^1\\ x\in K_{r(e)}}} \overline{\xi(\phi_e(x),x)}\eta(\phi_e(x),x) = \langle \xi, \eta\rangle_A(x),\end{aligned}$$

for all $\xi, \eta \in X$. Finally, let $\eta \in \mathcal{X}$. For $(y,x) \in \mathcal{G}$, since $\mathcal{M}$ satisfies the graph separation condition, there is a unique $e \in E^1$ such that $y = \phi_e(x)$. Define $\xi \in X$ by the formula $\xi(y,x) = \eta(e,x)$ for all $(y,x) \in \mathcal{G}$. Then ξ is well defined and $V(\xi) = \eta$, hence V is onto. Thus V is a C^*-correspondence isomorphism. Therefore $\mathcal{O}_{\mathcal{M}}(K)$ is isomorphic to $C^*(G)$. ■

***Example 4.3* (The two-part dust)** Let G be the graph from the figure



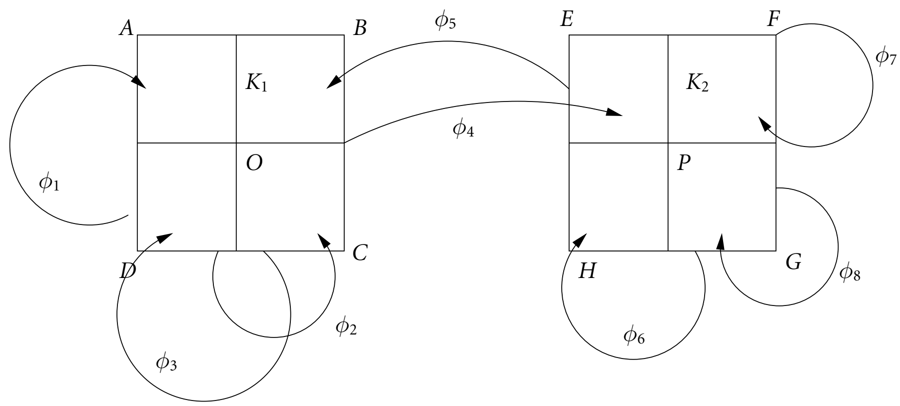
Let $T_1, T_2 \subset \mathbb{R}^2$ be two disjoint compact sets such that K_1 contains the origin, K_2 contains $(0,1)$, and K_1 and K_2 are symmetric about the x-axis. Let $\{\phi_i\}_{i=1,\ldots,4}$ be similarities, such that the map ϕ_1 has ratio $1/2$, fixed point $(0,0)$, and rotation 30 degrees counterclockwise. The map ϕ_2 has ratio $1/4$, fixed point $(1,0)$, and rotation 60 degrees clockwise. The map ϕ_3 has ratio $1/2$, fixed point $(0,0)$ and rotation 90 degrees counterclockwise. The map ϕ_4 has ratio $3/4$, fixed point $(1,0)$, and rotation 120 degrees clockwise (see [10, p. 167] for more details). Then the Mauldin–Williams graph $\mathcal{M}$ satisfies the graph separation condition, hence $\mathcal{O}_{\mathcal{M}}(K)$ is isomorphic to $C^*(G)$.

One can see that any Mauldin–Williams graph associated with this graph G will satisfy the graph separation condition, hence the associated Cuntz–Pimsner algebra will be isomorphic to $C^*(G)$.

Example 4.4 Let $G = (E^0, E^1, r, s)$ be the graph with the vertex matrix A_G

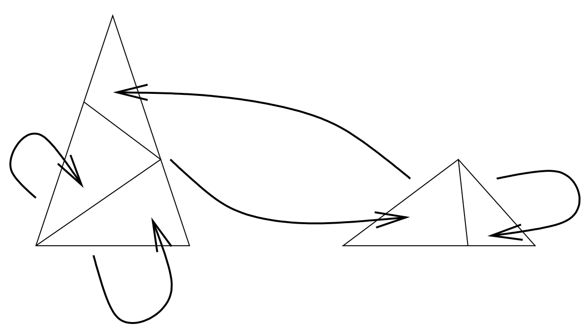
$$A_G = \begin{bmatrix} 3 & 1\\ 1 & 3\end{bmatrix}.$$

Hence $K_1(C^*(G)) = 0$ and $K_0(C^*(G)) \simeq \mathbb{Z}/3\mathbb{Z}$. Let K_1 and K_2 be two copies of the square of length one with the maps $\{\phi_i\}_{i=1,\dots,8}$ as in the following diagram:



That is, $\{\phi_i\}_{i=1,\dots,8}$ are similarities, $\phi_1, \phi_2, \phi_3 : K_1 \to K_1$, $\phi_4 : K_1 \to K_2$, $\phi_5 : K_2 \to K_1$, $\phi_6, \phi_7, \phi_8 : K_2 \to K_2$, such that $\phi_1(A) = A$, $\phi_1(C) = \phi_2(C) = O$, $\phi_2(A) = C$, $\phi_3(D) = D$, $\phi_3(B) = O$, $\phi_4(A) = E$, $\phi_4(C) = P$, $\phi_5(F) = B$, $\phi_5(H) = 0$, $\phi_6(F) = P$, $\phi_6(H) = H$, $\phi_7(F) = F$, $\phi_7(H) = P$, $\phi_8(E) = P$, $\phi_8(G) = G$. Then $K = K_1 \sqcup K_2$ is the invariant set of the Mauldin–Williams graph and $B(\mathcal{M}) = \{O\}$. So $A = C(K_1) \oplus C(K_2)$, $I_X = C_0(K_1 \setminus \{O\}) \oplus C(K_2)$. Then $K_1(\mathcal{O}_{\mathcal{M}}(K)) \simeq 0$ and $K_0(\mathcal{O}_{\mathcal{M}}(K)) \simeq \mathbb{Z}/2\mathbb{Z}$. Therefore $\mathrm{K}_0(\mathcal{O}_{\mathcal{M}}(K))$ is not isomorphic to $\mathrm{K}_0(C^*(G))$. Thus $\mathcal{O}_{\mathcal{M}}(K)$ is not isomorphic to $C^*(G)$.

Example 4.5 **(Penrose tiling)** The tiles of the Penrose tiling are two triangles such that the angles of the first triangle are equal to $\frac{\pi}{5}, \frac{2\pi}{5}, \frac{2\pi}{5}$ and the angles of the second triangle are equal to $\frac{3\pi}{5}, \frac{\pi}{5}, \frac{\pi}{5}$. These triangles are cut from a rectangular pentagon by the diagonals issued from a common vertex. Hence the ratio between the lengths of the shorter and the longer sides of the triangles is equal to $\tau = \frac{1+\sqrt{5}}{2}$. The Mauldin–Williams graph associated with the Penrose tiling is given in the following diagram, as shown in [3]:



Then the invariant set K of this Mauldin–Williams graph is the union of the two triangles. The vertex matrix of the graph is

$$A_G = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix},$$

hence $K_0(C^*(G)) = K_1(C^*(G)) = 0$. Since $K_0(\mathcal{O}_{\mathcal{M}}(K)) \simeq \mathbb{Z}$ and $K_1(\mathcal{O}_{\mathcal{M}}(K)) = 0$, $\mathcal{O}_{\mathcal{M}}(K)$ is not isomorphic to $C^*(G)$.

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