Exercise 10.14 (c, d and e)

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(c) We know that if G is not cyclic, then it must have an element of order 3; call it a.

Now, e, a and a^2 are unique by Thm 4.5. Fix b to be any element of G different from e, a, and a^2. Now b, ab and a^2b are distinct, for if $a^mb = a^nb$ for an n, m = 0, 1, or 2 then by $a^m = a^n$ and m = n by Thm 4.5. Finally the elements e, a, and a^2 are distinct from b, ab, and a^2b. To see this, just suppose $a^m = a^nb$. Then $b = a^{(m-n)}$. Reducing the exponent on a modulo 3 we see that $b = a^j$ for j = 0, 1, or 2. This contradicts the choice of b. Consequently G must be comprised of the six distinct elements e, a, a^2, b, ab, and a^2b.

(d) Given our assumptions on G and b, ord(b) = 2 or 3. Just suppose ord(b) = 3. Consider b^2. This must be one of the elements in G. It can't be b, ab, or a^2b, else b is e, a, or a^2. So b^2 is either e, a, or a^2. Since ord(b) = 3, then b^2 can't be e. If $b^2 = a$, then, $b^{-1} = b^2 = a$ and $b = a^{-1} = a^2$, which contradicts the choice of b above. So $b^2 = a^2$, or $b^{-1} = a^{-1}$. By the uniqueness of inverses, a = b. This final contradiction shows ord(b) = 2.

In a similar way, ord(ab) = ord (a^2b) = 2.

(e) Consider ba in G. This element can't be e, otherwise $b = a^{-1} = a^2$. It can't be a, else b = e. It can't be a^2 else b = a. Now if ba = b, then a = e. Also, if ba = ab, then $(ba)^2 = (ab)^2 = e$ [from above]. So $e = baba = b^2a^2 = a^2$. This contradicts the ord(a) = 3. So it must be that $ba = a^2b$.

In a similar way, one checks that $ba^2 = ab$.