Group 5

11.23

a) $N(H) = \{ g \in G \mid gHg^{-1} = H \}$

then $eHe^{-1} = H$, so $e \in N(H) \neq \phi$.

Let $a, b \in N(H)$.

Then $aHa^{-1} = H$ and $bHb^{-1} = H$, and by multiplying $H = a^{-1}Ha$.

$$\begin{aligned} \text{So } (ab)H(ab)^{-1} &= (ab)H(b^{-1}a^{-1}) \\ &= a(bHb^{-1})a^{-1} \\ &= aHa^{-1} = H, \text{ so } ab \in N(H). \end{aligned}$$

Also $a^{-1}H(a^{-1})^{-1} = a^{-1}Ha = H$, so $a^{-1} \in N(H)$.

Thus by thm 5.1, $N(H) \leq G$

b) Let $h \in H$.

So $hHh^{-1} \subseteq H$ because $H \leq G$.

For $h' \in H$, note $h' = h(h^{-1}h'h)h^{-1} \in hHh^{-1}$.

Therefore, $H = hHh^{-1}$. So $h \in N(H)$ and $H \subseteq N(H)$.

Now sHs^{-1} $\forall s \in N(H)$, so by thm 11.1, $H \triangleleft N(H)$.

c) Given $K \leq G$, $H \triangleleft K$.

By thm 11.1, $kHk^{-1} = H$ $\forall k \in K$,

so $k \in N(H)$ $\forall k \in K$, so $K \subseteq N(H)$.