Group Assignment 4

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13.13 Let $A \triangleleft G$ and $B \triangleleft H$. Must it be true that $\frac{G\times H}{A\times B} \simeq \frac{G}{A} \times \frac{H}{B}$?
Either prove that it must or give a counterexample.

Consider $\varphi : G\times H \to \frac{G}{A} \times \frac{H}{B}$
defined naturally by $\varphi(g, h) = (A\,g, B\,h)$
Clearly, φ is onto, as the domain of φ admits every g, h pair, which in turn define all possible pairs of right cosets.

Also, $\varphi(g_1\,g_2, h_1\,h_2) = (A\,g_1\,g_2, B\,h_1\,h_2) = (A\,g_1\,A\,g_2, B\,h_1\,B\,h_2) = (A\,g_1, B\;h_1)\,(A\,g_2, B\,h_2) = \varphi(g_1, h_1)\,\varphi(g_2, h_2)$
So φ is a homomorphism

Let $(g, h) \in A\times B$
so $(g, h)\,(e_G, e_H)^{-1} \in A\times B$ and thus
$(g\,e_G{}^{-1}, h\,e_H{}^{-1}) \in A\times B$
So $\varphi(g, h) = (A\,g, B\,h) = (A\,e_G, B\,e_H) = e_{\frac{G}{A}\times\frac{H}{B}}$
Thus $(g, h) \in \ker(\varphi)$

Let $(g, h) \notin A\times B$
So $(A\,g, B\,h) \neq (A\,e_G, B\,e_H)$
(since either $g \notin A$ and thus $A\,g \neq A\,e_G$, or $h \notin B$ and $B\,h \neq e_H$)
so $\varphi(g, h) = (A\,g, B\,h) \neq (A\,e_G, B\,e_H)$
So $(g, h) \notin \ker(\varphi)$

Thus $\ker(\varphi) = A\times B$
(note: we have just proven an auxilliary result, for $A \triangleleft G$ and $B \triangleleft H$: $A\times B \triangleleft G\times H$)

By the Fundamental theorem of homomorphisms:
$G\times H / \ker(\varphi) = \frac{G\times H}{A\times B} \simeq \frac{G}{A} \times \frac{H}{B}$