14.4 For n, a positive integer, every abelian group of order n is cyclic iff n is not divisible by the square of any prime.

(i.e. if n is divisible by the square of a prime, then there exist atleast one finite abelian group of order n which is not cyclic.)

($\Leftarrow$)

$$|G| = n = p_1 p_2 \cdots p_m \quad \text{since} \quad p_i^2 \nmid n, \ \forall i.$$

By Thm 14.2 / Thm 1:

$$G \cong G(p_1) \times G(p_2) \times \cdots \times G(p_m).$$

By Thm 6.1:

$G(p_1) \times G(p_2) \times \cdots \times G(p_m)$ is cyclic,

So $G \cong G(p_1) \times G(p_2) \times \cdots \times G(p_m)$ is cyclic.

($\Rightarrow$) By contrapositive, let $p_i^2 \mid n$.

So $n = p_1^j p_2^k \cdots p_m^\ell$ where atleast one exponent is greater than 1.

So $\forall\, G_x$ where $|G_x| = n$,

$$G_x \cong G_x(p_1^j) \times G_x(p_2^k) \times \cdots \times G_x(p_i^\ell) \times \cdots \times G_x(p_m^\ell)$$

and there exists a G_x and a partition $\pi \in \mathbb{P}(\ell)$ (Thm 2)

where $\pi = (1, 1, \ldots, 1)$

and $$G_x \cong \underbrace{G_x(p_i) \times G_x(p_i) \times \cdots \times G_x(p_i)}_{\ell \text{ times}} \times G_x(p_1^j) \times G_x(p_2^k) \times \cdots \times G_x(p_m^\ell)$$

By Thm 6.1, G_x is not cyclic.

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