Exercise 17.21
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Let $R$ denote a ring with $Z(R) = \{ r \in R \mid rs = sr \text{ for all } s \in R \}$.

$Z(R)$ is a subring of $R$. To see this, fix $r, r' \in Z(R)$ and $s \in R$. We see $(r - r')s = rs - r's = sr - sr' = s(r - r')$. Thus $r - r' \in Z(R)$. Also, $(rr')s = r(r's) = r(sr') = (rs)r' = s(rr')$. So, $rr' \in Z(R)$. Since these two properties hold for $Z(R)$, we have $Z(R)$ is a subring of $R$.

However $Z(R)$ need not be an ideal in $R$. To see this consider $R = M(2, \mathbb{Z})$.

Consider $Z = 2I$ where $I$ is the identity matrix in $R$. Clearly this matrix is in $Z(R)$ as it commutes with all $2 \times 2$ integer matrices. Consider the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

One checks that $ZA$ is not in $Z(R)$ as it doesn’t commute with, for example, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. 