Exercise 17.21

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Let R denote a ring with $Z(R) = \{ r \in R \mid rs = sr \text{ for all } s \in R \}$.

<u>Z(R) is a subring of R</u>. To see this, fix $r, r' \in Z(R)$ and $s \in R$. We see $(r - r')s = rs - r's = sr - sr' = s(r - r')$. Thus $r - r' \in Z(R)$. Also, $(rr')s = r(r's) = r(sr') = (rs)r' = s(rr')$. So, $rr' \in Z(R)$. Since these two properties hold for Z(R), we have Z(R) is a subring of R.

However <u>Z(R) need not be an ideal</u> in R. To see this consider $R = M(2,\mathbf{Z})$. Consider $Z = 2I$ where I is the identity matrix in R. Clearly this matrix is in Z(R) as it commutes with all 2 x 2 integer matrices. Consider the matrix A =

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

One checks that ZA is not in Z(R) as it doesn't commute with, for example,

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$