Group Assignment 1 — Brian Dinneen, Heather Clark, Shirley Han, Kellen Myers

2.8) Let G be the set of all real-valued functions f on the real line which haev the property that $f(x) \neq 0$ for all $x \in \mathbb{R}$. Define the product $f \times g$ of two functions f, g in G by
$(f \times g)(x) = f(x)\,g(x)$ for all $x \in \mathbb{R}$.
With this operation, goes G form a group? Prove or disprove.

Yes, we will show that $(G, \times)$ satisfies all four conditions of being a group.

i) G is a set and $\times$ is a binary operation on G

G is defined as a set.
$(f \times g)(x) = f(x)\,g(x)$ is a well-defined function on $\mathbb{R}$.
Thus, to show $f \times g \in G$ we must verify that $(f \times g)(x) \neq 0$ for all $x \in \mathbb{R}$.
Assume otherwise, that is, there is an $x_0 \in \mathbb{R}$ such that $(f \times g)(x_0) = f(x_0)\,g(x_0) = 0$
Thus, $f(x_0) = 0$ or $g(x_0) = 0$, contradicting $f, g \in G$.

ii) $\times$ is associative

$((f \times g) \times h)(x) = (f(x)\,g(x))\,h(x) = f(x)\,(g(x)\,h(x)) = (f \times (g \times h))(x)$
This fact follows from the associativity of real numbers under multiplication.

iii) There is an identity element e where $e \times f = f \times e = f$ for any $f \in G$

Define $e(x) = 1$; for any $g \in G$
$(g \times e)(x) = g(x) \cdot 1 = g(x) = 1 \cdot g(x) = (e \times g)(x)$

iv) For each $f \in G$ there is an f^{-1} such that $f \times f^{-1} = f^{-1} \times f = e$

Define $f^{-1}(x) = \frac{1}{f(x)}$ for all $x \in \mathbb{R}$. f^{-1} is well defined on $\mathbb{R}$ because $0 \notin f(\mathbb{R})$
$(f^{-1} \times f)(x) = f(x)\,\frac{1}{f(x)} = 1 = e = \frac{1}{f(x)}\,f(x) = (f \times f^{-1})(x)$