Group 1

(6.10) find all subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_4$

$\mathbb{Z}_2 = \{0,1\}$ $\mathbb{Z}_4 = \{0,1,2,3\}$

subgroups of order 1: $\{(0,0)\}$

subgroups of order 2: $\{(0,0),(1,0)\}$; $\{(0,0),(1,2)\}$

$\{(0,0),(0,2)\}$

subgroups of order 4: $\{(0,0),(0,1),(0,2),(0,3)\}$

$\{(0,0),(1,1),(0,2),(1,3)\}$

$\{(0,0),(1,0),(0,2),(1,2)\}$

subgroup of order 8: $\{(0,0),(0,1),(0,2),(0,3),(1,0),(1,1),$
$,(1,2),(1,3)\} = \mathbb{Z}_2 \times \mathbb{Z}_4$

we will show that there are no subgroups of order 3, 5, 6 or 7

by theorem 5.5, all subgroups of $\mathbb{Z}_4$ have order 0, 2, 4
all subgroups of $\mathbb{Z}_2$ have order 0, 2

suppose the exists a subgroup of $\mathbb{Z}_2 \times \mathbb{Z}_4$ of order 3, 5 or 7:
if the first element of the carthesian pair will go back to itself, then the second element would have to have order 3, 5 or 7: impossible

if the ~~second~~ first element of the carthesian pair will have order 2, then 2 will have to divide the number of elements of the subgroup : impossible

Suppose there exists a subgroup of order 6 of $\mathbb{Z}_2 \times \mathbb{Z}_4$
if the first element of the carthesian pair will have order 0, then the second ~~order~~ elem. would have to be order 6 which is impossible in $\mathbb{Z}_4$

if the first element of the pair will have order 2, then 2 has to divide the number of elem. of the subgroup, which is true

then the second element would have to have either order 3 or order 6:

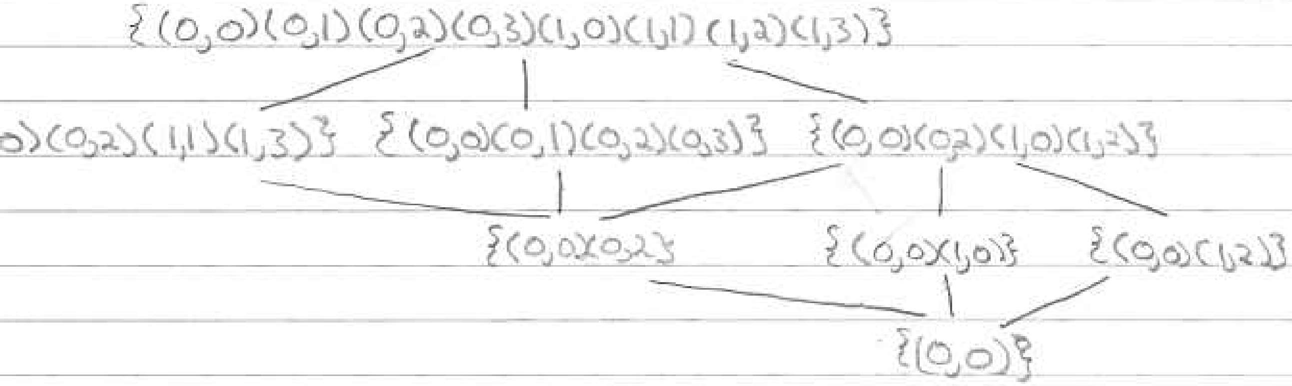
because then we'd have the first element generate two different pairs for each new element of the second part of the pair

in which case the second position of the pair would generate all the elements of the subgroup

however, the second element cannot have order 6 in $\mathbb{Z}_4$ and also it cannot have order 3 by theorem 5.5. $\times$

therefore, we have shown that there are no subgroups of order 3, 5, 6, 7

so we have listed all the subgroups of $\mathbb{Z}_2$ and $\mathbb{Z}_4$


{(0,0)(0,1)(0,2)(0,3)(1,0)(1,1)(1,2)(1,3)}
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