9.13 – <u>Group 4: K Hwang, L Jones, L Kenny, A Leeman, C Ozor</u>

Let H be a subgroup of G and define ${}_H\!\sim$ on G by $x\,{}_H\!\sim y$ iff $x^{-1}y \in H$.

a) Show that ${}_H\!\sim$ is an equivalence relation on G.

i. Let $x \in G$. (W.T.S.: Reflexivity)
Since $x^{-1}x = e$ and $e \in H$, we know $x^{-1}x \in H$ and $x\,{}_H\!\sim x$.

ii. Let $x\,{}_H\!\sim y$. (W.T.S.: Symmetry)
Then $x^{-1}y \in H$.
We know $(x^{-1}y)^{-1} \in H$.
So it then follows that $y^{-1}x \in H$.

iii. Fix $x,y,z \in G$ with $x\,{}_H\!\sim y$ and $y\,{}_H\!\sim z$. (W.T.S.: Transitivity)
So, $x^{-1}y \in H$ and $y^{-1}z \in H$.
Then $(x^{-1}y)(\,y^{-1}z) = x^{-1}z \in H$.
Therefore, $x\,{}_H\!\sim z$.

* So ${}_H\!\sim$ is an *equivalence relation* on G because all three properties (reflexivity, symmetry, and transitivity) hold.

b) Show that the equivalence clas~~s~~es under ${}_H\!\sim$ are the left cosets of H in G.

We claim: $xH = \{y \in G \mid y\,{}_H\!\sim x\} = \bar{x}$. From this we want to show that the equivalence class under ${}_H\!\sim$ are the left cosets of H in G.

Fix $y \in \bar{x}$.
Since $x\,{}_H\!\sim y$ then $x^{-1}y = h$ for some $h \in H$.
Then it follows that $y = xh$.
So, $\{y \in G \mid y\,{}_H\!\sim x\} \subseteq \{xh \mid h \in H\} = xH$. [We are now halfway done.]

Next, fix $xh' \in xH$ where $h' \in H$.
Consider $x^{-1}(xh') = h' \in H$.
So $x\,{}_H\!\sim xh'$.
Then $xh' \in \bar{x}$.
Thus, $xH = \bar{x} = \{y \in G \mid y\,{}_H\!\sim x\}$.

* Therefore, the equivalence clas~~s~~es under ${}_H\!\sim$ are the left cosets of H in G.