Let \( H \) be a subgroup of \( G \) and define \( \sim \) on \( G \) by \( x \sim y \) iff \( x^{-1}y \in H \).

a) Show that \( \sim \) is an equivalence relation on \( G \).

i. Let \( x \in G \). (W.T.S.: Reflexivity)
Since \( x^{-1}x = e \) and \( e \in H \), we know \( x^{-1}x \in H \) and \( x \sim x \).

ii. Let \( x \sim y \). (W.T.S.: Symmetry)
Then \( x^{-1}y \in H \).
We know \( (x^{-1}y)^{-1} \in H \).
So it then follows that \( y^{-1}x \in H \).

iii. Fix \( x, y, z \in G \) with \( x \sim y \) and \( y \sim z \). (W.T.S.: Transitivity)
So, \( x^{-1}y \in H \) and \( y^{-1}z \in H \).
Then \( (x^{-1}y)(y^{-1}z) = x^{-1}z \in H \).
Therefore, \( x \sim z \).

* So \( \sim \) is an equivalence relation on \( G \) because all three properties (reflexivity, symmetry, and transitivity) hold.

b) Show that the equivalence classes under \( \sim \) are the left cosets of \( H \) in \( G \).

We claim: \( xH = \{ y \in G | y \sim x \} = \bar{x} \). From this we want to show that the equivalence class under \( \sim \) are the left cosets of \( H \) in \( G \).

Fix \( y \in \bar{x} \).
Since \( x \sim y \) then \( x^{-1}y = h \) for some \( h \in H \).
Then it follows that \( y = xh \).
So, \( \{ y \in G | y \sim x \} \subseteq \{ xh | h \in H \} = xH \). [We are now halfway done.]

Next, fix \( xh' \in xH \) where \( h' \in H \).
Consider \( x^{-1}(xh') = h' \in H \).
So \( x \sim xh' \).
Then \( xh' \in \bar{x} \).
Thus, \( xH = \bar{x} = \{ y \in G | y \sim x \} \).

* Therefore, the equivalence classes under \( \sim \) are the left cosets of \( H \) in \( G \).