September 28, 2004

Math 111 C — Exam I

Show all work clearly for partial credit. Do not use the graphing capabilities of your calculator.

1. *(8 points)* Find exactly; i.e., do not use a calculator:

   (a) $\log_2 10 + 2\log_2 6 - \log_2 45$ (b) $\dfrac{\ln \sqrt[5]{125}}{\ln 5}$

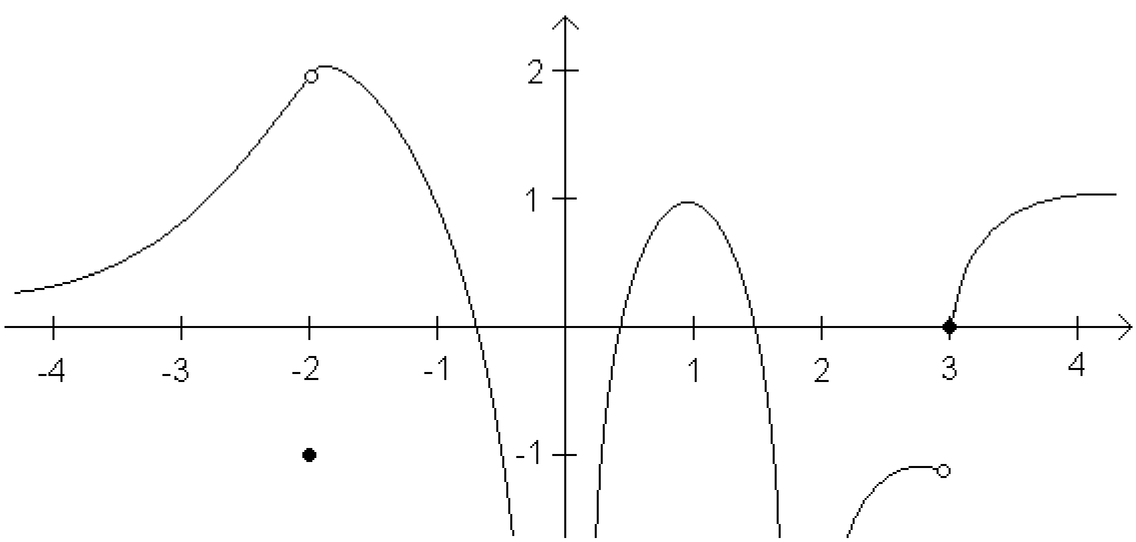
2. *(20 points)* Find the limits, if they exist:

   (a) $\displaystyle\lim_{x\to 1} \frac{x^2-3x+2}{x^2-1}$ (b) $\displaystyle\lim_{x\to 2} \frac{x^2+3x+2}{x^2-1}$

   (c) $\displaystyle\lim_{x\to 0} \ln(x^2)$ (d) $\displaystyle\lim_{x\to 0^-} \left(\frac{1}{|x|}+\frac{1}{x}\right)$

3. *(14 points)* For the function $f(x)$ with the graph below, find or approximate (if they exist):

   (a) $f(-2)$,
   (b) $\lim_{x\to -2} f(x)$,
   (c) the equation(s) of the vertical asymptote(s),
   (d) the equation(s) of the horizontal asymptote(s),
   (e) $\lim_{x\to 0} f(x)$,
   (f) the x-value(s) at which f has a removable discontinuity, and
   (g) the slope of the tangent line to the graph of $y = f(x)$ at $x = 1$.



4. *(18 points)* Find the equations of the horizontal and vertical asymptotes:

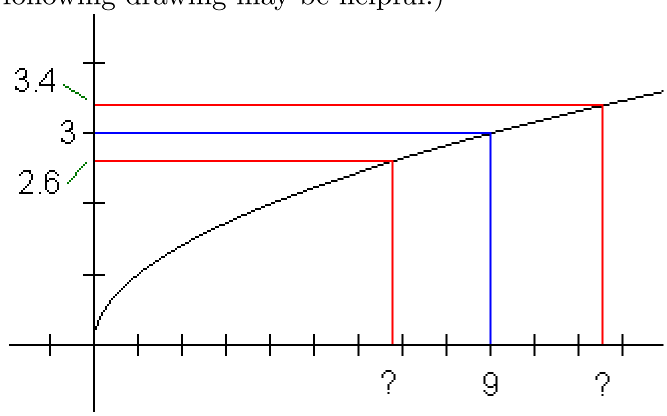
   (a) $f(x) = \dfrac{x^2-4}{x^2-3x+2}$ (b) $g(x) = \dfrac{2x}{\sqrt{x^2+4}}$

5. *(17 points)* (a) Write down in terms of h an expression for the slope of the (secant) line joining the points on the graph of the function $f(x) = \sqrt{2x+3}$ with the x-values $x = 11$ and $x = 11 + h$.
(b) Using your answer to (a), find the slope of the tangent line to the graph of the function $f(x) = \sqrt{2x+3}$ at the point $x = 11$. (Use of a derivative formula — whatever that is — or simply the answer will receive no credit. Note: This is the same as the instantaneous rate of change of $f(x) = \sqrt{2x+3}$ with respect to x at the point where $x = 11$.)
(c) Using your answer to (b), write the equation of the tangent line to the graph of $f(x) = \sqrt{2x+3}$ at $x = 11$.

6. *(15 points)* Let $f(x) = \sqrt{x}$, $a = 9$ and $\varepsilon = 0.4$. What is the largest value of δ for which,

$$\text{if } 0 < |x - 9| < \delta \text{ , then } |\sqrt{x} - 3| < .4 \text{ ?}$$

(An answer of the form "the smaller of the two numbers ________ and ________" is preferred but not required. The following drawing may be helpful.)



7. *(8 points)* The area of an algae growth on the surface of the liquid in a vat doubles every 10 hours. At midnight the area was 4 cm^2. Write a formula in terms of t for the area $A(t)$ of the algae growth t hours after midnight.

Some possibly useful equations:

$$y - y_0 = m(x - x_0) \qquad a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

$$a^r = b \Longleftrightarrow \log_a b = r \qquad \log_a c = \frac{\log_b c}{\log_b a}$$

Solutions to Math 111C Exam I

1. (a) $\log_2(10(6^2)/45) = \log_2(360/45) = \log_2 8 = \log_2 2^3 = 3$.
   (b) $\log_5(125^{1/5}) = \log_5(5^3)^{1/5} = \log_5(5^{3/5}) = 3/5$.

2. (a) $\lim_{x\to 1}((x-2)(x-1))/((x-1)(x+1)) = \lim_{x\to 1}(x-2)/(x+1) = (1-2)/(1+1) = -1/2$.
   (b) The denominator does not approach 0 as $x \to 2$, so we can find the limit by substitution: $(2^2 + 3(2) + 2)/(2^2 - 1) = 4$.
   (c) As $x \to 0$ from either the right or left, x^2 approaches 0 from the right, so its natural logarithm approaches $-\infty$. (To say that the limit does not exist is also acceptable, because $-\infty$ is not a real number.)
   (d) For all x-values less than 0, we have $|x| = -x$, so $(1/|x|) + (1/x) = 0$, so the limit is 0.

3. (a) -1 (as nearly as we can tell), the location of the solid dot.
   (b) 2 (as nearly as we can tell), the value toward which the graph is heading.
   (c) $x = 0$ (the y-axis) and $x = 2$.
   (d) $y = 0$ (the x-axis) to the left and $y = 1$ to the right.
   (e) $-\infty$. (To say that the limit does not exist is also acceptable, because $-\infty$ is not a real number.)
   (f) $x = -2$. The rest are jump discontinuities or worse.
   (g) 0 (the tangent looks horizontal).

4. (a) $\lim_{x\to\infty}(1 - (4/x^2))/(1 - (3/x) + (2/x^2)) = (1-0)/(1-0+0) = 1$, and this is also the limit as $x \to -\infty$, so $y = 1$ is a horizontal asymptote to both the right and left. Because $(x^2 - 4)/(x2 - 3x + 2) = (x+2)/(x-1)$ except at $x = 2$, there is a removable discontinuity at $x = 2$; the only vertical asymptote is $x = 1$, where the denominator is 0 and the numerator is not.
   (b) Because the denominator is never 0, there are no vertical asymptotes. Now:

$$\begin{aligned}\lim_{x\to\infty}\frac{2x}{\sqrt{x^2+4}} &= \lim_{x\to\infty}\frac{2}{\sqrt{1+(4/x^2)}} = \frac{2}{\sqrt{1+0}} = 2\\ \lim_{x\to-\infty}\frac{2x}{\sqrt{x^2+4}} &= \lim_{x\to-\infty}\frac{2}{-\sqrt{1+(1/x^2)}} = -\frac{2}{\sqrt{1+0}} = -2\end{aligned}$$

   So $y = 2$ is a horizontal asymptote to the right and $y = -2$ is one to the left.

5. (a)

$$\frac{\sqrt{2(11+h)+3} - \sqrt{2(11)+3}}{h}$$

   (b)

$$\begin{aligned}\lim_{h\to 0}\frac{\sqrt{2(11+h)+3} - \sqrt{2(11)+3}}{h} &= \lim_{h\to 0}\frac{(2(11+h)+3) - (2(11)+3)}{h(\sqrt{2(11+h)+3} + \sqrt{2(11)+3})}\\ &= \lim_{h\to 0}\frac{2}{\sqrt{2(11+h)+3} + \sqrt{2(11)+3}}\\ &= \frac{2}{\sqrt{2(11+0)+3} + \sqrt{2(11)+3}}\\ &= \frac{2}{2\sqrt{2(11)+3}} = \frac{1}{5}\end{aligned}$$

(c) Because $f(11) = 5$, the desired tangent line is $y - 5 = \frac{1}{5}(x - 11)$.

6. Because the inverse of the function $y = \sqrt{x}$ is $x = y^2$, the question marks are 2.6^2 and 3.4^2; so the answer is: the largest δ that works is the smaller of the two numbers $9 - 2.6^2 [= 2.24]$ and $3.4^2 - 9 [= 2.56]$, [i.e., 2.24]. It would not have been necessary to give any of the material in the brackets.

7. This is an example of "exponential growth", so the formula is $A(t) = A_0(2^{kt})$, where A_0 is the area at time 0 (which we are told is 4) and the "growth rate" k is a constant chosen to fit the given information. (We could have used a different base in place of 2, and then k would have been different.) In this case, because we know that, when $t = 10$, $A(10)$ is twice as large as it was at midnight, so it is 8: $8 = 4(2^{k(10)})$, so $1 = k(10)$, so $k = 1/10$. So the formula is $A(t) = 4(2^{t/10})$.