October 19, 2004

Math 111 C — Exam II

Show all work clearly for partial credit. Do not use the graphing capabilities of your calculator.

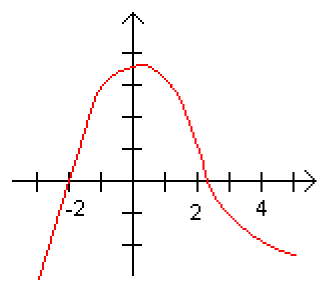
1. *(42 points)* Find the derivatives:

   (a) $g(x) = 5x^8 - 2x^5 + 6$ (b) $u = \sqrt[3]{t^2} + 2\sqrt{t^3}$ (c) $y = 2^{\sin(\pi x)}$

   (d) $y = e^u(\cos u + 4u)$ (e) $F(z) = \sqrt{\dfrac{z-1}{z+1}}$

2. *(13 points)* For $f(x) = \sqrt{1+2x}$, find $f'(x)$ using the limit definition of the derivative. Use of derivative formulas or simply the answer will receive no credit.

3. *(12 points)* (a) Find dy/dx if $y = x/(1+x^2)$.
   (b) Find an equation of the tangent line to the curve $y = x/(1+x^2)$ at the point where $x = 3$.



4. *(8 points)* For the function g whose graph is given, arrange the following numbers in increasing order:

   $$0, \quad g'(-2), \quad g'(0), \quad g'(2), \quad g'(4)$$

5. *(10 points)* A stone is dropped into a lake, creating a circular ripple that travels outward at a speed of 12 cm/sec. Find the rate at which the area within the circle is increasing t sec after the stone is dropped, in terms of t. (Hint: Express the radius of the circle in terms of t.)

6. *(15 points)* Recall that the number e was chosen so that the slope of $y = e^x$ at $(0, 1)$ is 1.

   (a) What is the slope of $y = \ln x$ at $(1, 0)$? (Hint: From the graph of $y = e^x$, how do you get the graph of $y = \ln x$?)

   (b) In view of your answer to (a), what is

   $$\lim_{t \to 0} \frac{\ln(1+t)}{t} \ ?$$

   (Hint: How do we write down the slope of $y = \ln x$ at $(1, 0)$ as a limit?)

   (c) Use the limit definition of derivative, the substitution $h/x = t$, and your answer to (b) to prove that

   $$\frac{d}{dx}(\ln x) = \frac{1}{x} \ .$$

Some possibly useful equations:

$$V = e^3 \qquad A = \pi r^2 \qquad A = 6e^2$$

$$y - y_0 = m(x - x_0) \qquad h = h_0 + v_0 t - \frac{1}{2} g t^2$$

Solutions to Math 111C Exam II

1. (a) $g'(x) = 40x^7 - 10x^4$ (b) $du/dt = \frac{2}{3}t^{-1/3} + 3t^{1/2}$

   (c) $dy/dx = 2^{\sin \pi x}(\ln 2)(\cos \pi x)\pi$ (d) $dy/du = e^u(-\sin u + 4) + (\cos u + 4u)e^u$

   (e)
$$F'(z) = \frac{1}{2}\left(\frac{z-1}{z+1}\right)^{-1/2}\left(\frac{(z+1)-(z-1)}{(z+1)^2}\right) = \frac{1}{2}\sqrt{\frac{z+1}{z-1}}\left(\frac{2}{(z+1)^2}\right)$$

2.
$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{\sqrt{1+2(x+h)}-\sqrt{1+2x}}{h} = \lim_{h\to 0}\frac{(1+2(x+h))-(1+2x)}{h(\sqrt{1+2(x+h)}+\sqrt{1+2x})} \\ &= \lim_{h\to 0}\frac{2}{\sqrt{1+2(x+h)}+\sqrt{1+2x}} = \frac{1}{\sqrt{1+2x}} \end{aligned}$$

3. (a) $dy/dx = [(1+x^2) - x(2x)]/(1+x^2)^2 = (1-x^2)/(1+x^2)^2$
   (b) When $x = 3$, $y = 3/(1+3^2) = 0.3$ and $dy/dx = (1-3^2)/(1+3^2)^2 = -0.08$, so the requested tangent line is $y - 0.3 = -0.08(x-3)$.

4. The slopes of the given graph at -2 and 0 are positive, with the slope at -2 larger than the slope at 0; and the slopes at 2 and 4 are negative, with the slope at 2 more negative than the slope at 4. So the desired arrangement is: $g'(2) < g'(4) < 0 < g'(0) < g'(-2)$.

5. Let r denote the radius of the ripple in cm; then $r = 12t$. We know that, if A denotes the area in cm^2 enclosed by the ripple, then $A = \pi r^2$, so $A = \pi(12t)^2 = 144\pi t^2$. So we get $dA/dt = 288\pi t$ cm^2/sec.

6. (a) Because the functions $y = e^x$ and $y = \ln x$ are inverses of each other, the graph of $y = \ln x$ is the result of flipping the graph of $y = e^x$ around the line $y = x$, so the slope of $y = \ln x$ at $(1,0)$ is also 1.

   (b) Using the fact that $\ln 1 = 0$, we see that the slope of $y = \ln x$ at $(1,0)$ is
$$\lim_{h\to 0}\frac{\ln(1+h)-\ln 1}{h} = \lim_{h\to 0}\frac{\ln(1+h)}{h} = \lim_{t\to 0}\frac{\ln(1+t)}{t},$$
   so by (a) the requested limit is 1.

   (c) Using (b) and the fact that, as h approaches 0, $t = h/x$ also approaches 0, we get
$$\begin{aligned} \frac{d}{dx}(\ln x) &= \lim_{h\to 0}\frac{\ln(x+h)-\ln x}{h} = \lim_{h\to 0}\frac{\ln\left(\frac{x+h}{x}\right)}{h} = \lim_{h\to 0}\frac{\ln(1+\frac{h}{x})}{h} \\ &= \lim_{h\to 0}\frac{\ln(1+t)}{xt} = \lim_{h\to 0}\frac{\ln(1+t)}{t}\cdot\frac{1}{x} = 1\cdot\frac{1}{x} = \frac{1}{x} \end{aligned}$$