September 28, 1999

Math 111 E and H — Exam I

Show all work clearly for partial credit. Do not use the graphing capabilities of your calculator.

1. *(8 points)* Find exactly; i.e., do not use a calculator:

   (a) $\log_3 45 - \log_3 5$ (b) $\dfrac{\log_2 \sqrt{27}}{\log_2 3}$

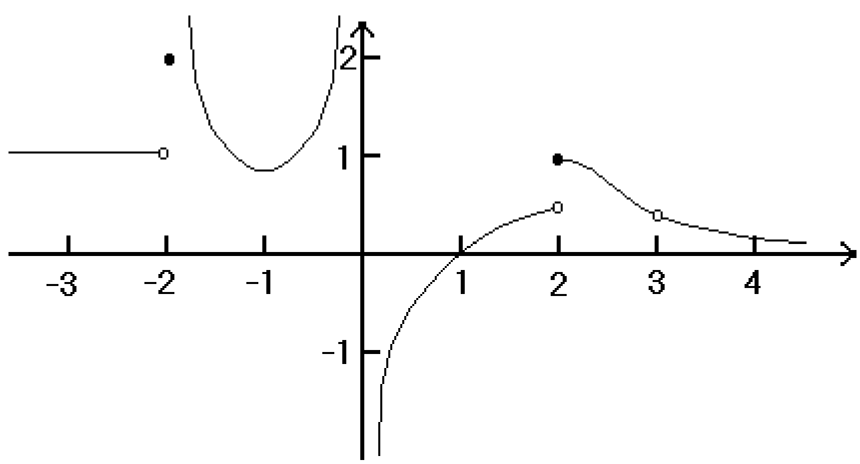
2. *(20 points)* Find the limits, if they exist:

   (a) $\lim_{x\to 3} \dfrac{x^2-3x+2}{x^2-4}$ (b) $\lim_{x\to 2} \dfrac{x^2+3x+2}{x^2-4}$

   (c) $\lim_{x\to\infty} (2e^{-x}-1)$ (d) $\lim_{x\to 0^-} \left(\dfrac{1}{x} - \dfrac{1}{|x|}\right)$

3. *(14 points)* For the function $f(x)$ with the graph below, find or approximate (if they exist):

   (a) $\lim_{x\to -2^-} f(x)$,
   (b) the equation(s) of the vertical asymptote(s),
   (c) the equation(s) of the horizontal asymptote(s),
   (d) $\lim_{x\to 0} f(x)$,
   (e) the x-value(s) at which f has a removable discontinuity,
   (f) $f'(-1)$ and
   (g) $f'(1)$



4. *(18 points)* Find the equations of the horizontal and vertical asymptotes:

   (a) $f(x) = \dfrac{x^2-1}{x^2-3x+2}$ (b) $g(x) = \dfrac{x}{\sqrt{x^2+1}}$

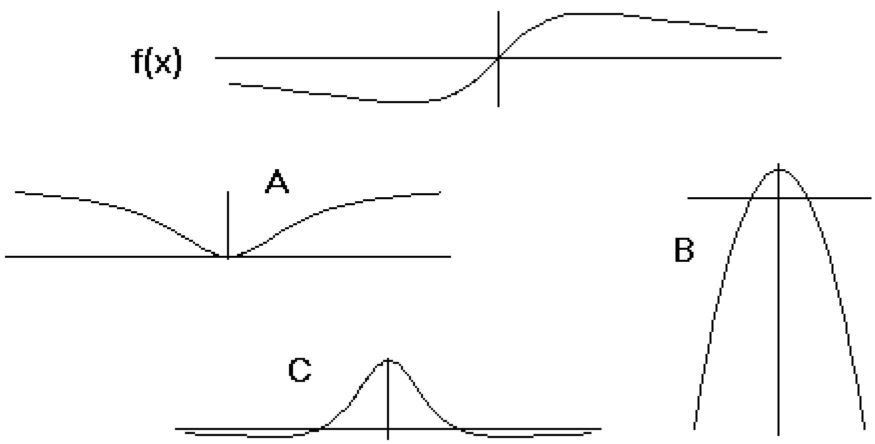
5. *(15 points)* (a) Use the limit definition of the derivative (in either form) to find $f'(11)$, the derivative of $f(x) = \sqrt{2x+3}$ at $a = 11$. Use of a derivative formula or simply the answer will receive no credit.
(b) Write the equation of the tangent line to $y = \sqrt{2x+3}$ at $x = 11$.

6. *(15 points)* Let $f(x) = x^2$, $a = 2$ and $\varepsilon = 0.5$. What is the largest value of δ for which,

$$\text{if } 0 < |x-2| < \delta \text{ , then } |x^2 - 4| < .5 \text{ ?}$$

(An answer of the form "the smaller of the two numbers ________ and ________" is preferred but not required.)

6. *(18 points)* Which of the lettered graphs A, B or C is that of the derivative of $f(x)$? Explain your answer.



Some possibly useful equations:

$$y - y_0 = m(x - x_0) \qquad a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

$$a^r = b \Longleftrightarrow \log_a b = r \qquad \log_a c = \frac{\log_b c}{\log_b a}$$

Solutions to Math 111EH Exam I

1. (a) $\log_3(45/5) = \log_3 9 = 2$.
   (b) $\log_3 \sqrt{27} = \log_3(3^3)^{1/2} = 3/2$.

2. (a) $\lim_{x\to 2}((x-2)(x-1))/((x-2)(x+2)) = \lim_{x\to 2}(x-1)/(x+2) = (2-1)/(2+2) = 1/4$.
   (b) $\lim_{x\to 2}((x+2)(x+1))/((x-2)(x+2)) = \lim_{x\to 2}(x+1)/(x-2)$, which does not exist: as x approaches 2 from the left, the quotient approaches $-\infty$, while as x approaches 2 from the right, the quotient approaches ∞.
   (c) As x grows without bound, $-x$ approaches $-\infty$, so e^{x} approaches 0; and hence the required limit is $2(0) - 1 = -1$.
   (d) For x-values less than 0, we have $|x| = -x$, so $(1/x) - (l/|x|) = 2/x$, which approaches $-\infty$ as x approaches 0 from the left.

3. (a) 1 (as nearly as we can tell).
   (b) $x = -2$ and $x = 0$ (the y-axis).
   (c) $y = 1$ to the left and $y = 0$ (the x-axis) to the right.
   (d) The limit does not exist. The two one-sided limits are ∞ and $-\infty$, so they do not agree.
   (e) $x = 3$. The rest are jump discontinuities or worse.
   (f) 0 and (g) about 1.

4. (a) $\lim_{x\to\infty}(1-(1/x^2))/(1-(3/x)+(2/x^2)) = (1-0)/(1-0+0) = 1$, and this is also the limit as $x \to -\infty$, so $y = 1$ is a horizontal asymptote to both the right and left. Because $(x^2 - l)/(x2 - 3x + 2) = (x + l)/(x - 2)$ except at $x = 1$, there is a removable discontinuity at $x = 1$; the only vertical asymptote is $x = 2$.
   (b) Because the denominator is never 0, there are no vertical asymptotes. Now:

$$\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{1}{\sqrt{1+(1/x^2)}} = \frac{1}{\sqrt{1+0}} = 1$$

$$\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to-\infty} \frac{1}{-\sqrt{1+(1/x^2)}} = -\frac{1}{\sqrt{1+0}} = -1$$

   So $y = 1$ is a horizontal asymptote to the right and $y = -1$ is one to the left.

5. (a)

$$\begin{aligned} f'(11) &= \lim_{h\to 0} \frac{\sqrt{2(11+h)+3} - \sqrt{2(11)+3}}{h} = \lim_{h\to 0} \frac{(2(11+h)+3) - (2(11)+3)}{h(\sqrt{2(11+h)+3} + \sqrt{2(11)+3})} \\ &= \lim_{h\to 0} \frac{2}{\sqrt{2(11+h)+3} + \sqrt{2(11)+3}} = \frac{2}{\sqrt{2(11+0)+3} + \sqrt{2(11)+3}} \\ &= \frac{2}{2\sqrt{2(11)+3}} = \frac{1}{5} \end{aligned}$$

   (b) Because $f(11) = 5$, the desired tangent line is $y - 5 = \frac{1}{5}(x - 11)$.

6. Because the inverse of the function $y = x^2$ is $x = \sqrt{y}$, at least on the part of the curve in which we are interested, the answer is: the largest δ that works is the smaller of the two numbers $2 - \sqrt{3.5}$ and $\sqrt{4.5} - 2$.

7. C: The tangents to $f(x)$ have small negative slopes both to the left and to the right, and in between the slopes become positive, with the greatest slope at x = 0.