December 14, 1999

Math 111 E and H — Final Exam

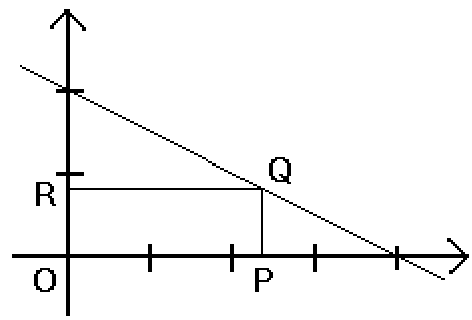
Show all work clearly for partial credit. Do not use the graphing capabilities of your calculator.

1. *(18 points)* Find:

   (a) $\dfrac{d}{dx}(\sin 3x \cos 2x)$ (b) $\displaystyle\int (\sqrt{x} - 4e^x)\,dx$ (c) $\displaystyle\lim_{x\to-\infty} \frac{x-1}{\sqrt{x^2+x-2}}$

2. *(10 points)* If $f'(x) = x^4 - 4x^3$, find intervals of concavity and inflection points of the original function f.

3. *(10 points)* Having come to Earth to establish trade, a squid-like creature is testing a Ferrari. Skillfully using three of its 10 tentacles for the brake, clutch and accelerator, it attains a constant acceleration. It is going 24 mi/hr as it passes a police car at a point 20 miles from its spaceship, and it arrives back at the ship 40 minutes (i.e., 2/3 hour) later. What was its constant acceleration? (Include units.)



4. *(10 points)* For Q on the line $y = 2 - \frac{1}{2}x$ in the first quadrant, what are the (a) maximum and (b) minimum areas of the rectangle $OPQR$? In particular, is the maximum rectangular area attained by a square?

5. *(10 points)* A kite is 40 feet above the ground and being carried $\frac{1}{2}$ ft/sec horizontally by the wind. The child holding the string is lying on the ground; how fast is the string coming off the spool in her hand when there are 50 feet of string out?

6. *(25 points)* Evaluate:
   (a) the average value of $3x^2 + 5$ on the interval $[-1, 2]$

   (b) $\displaystyle\int_0^{\pi/2} \frac{\cos x}{1+\sin x}\,dx$ (c) $\displaystyle\int_0^{\sqrt{3}} \frac{dx}{x^2+1}$ (d) $\displaystyle\int_{-0.2}^{0.2} \sin x^3\,dx$

7. *(7 points)* A shelf one foot deep is equipped with pressure sensors every 3 inches (1/4 foot) along its 2-foot length, including sensors at both ends, and then sand is dumped unevenly on it. The 9 sensors reveal pressures of 0, 1, 2, 2, 3, 3, 4, 2, and 1 pounds per square foot; and because the shelf is one foot deep, these figures are also pressures per linear foot of shelf length. Use the Midpoint Rule to approximate the total weight on the shelf.

8. *(5 points)* Display a function $F(x)$ for which $F'(x) = \sqrt{x^3+4}$ and $F(1) = 0$.

9. *(5 points)* For $x < 0$, find $D(\ln(-x))$.

Some possibly useful equations:

$a^2+b^2 = c^2$ $\qquad s(t) = \frac{1}{2}gt^2+v_0t+s_0$ $\qquad \dfrac{d}{dx}(\arctan u) = \dfrac{1}{1+u^2}$ $\qquad \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$

Solutions to Math 111EH Final Exam

1. (a) $\sin 3x(-\sin 2x)2 + (\cos 3x)3\cos 2x$
   (b) $\frac{2}{3}x^{3/2} - 4e^x + C$
   (c) Because we are considering negative x-values, we have $x = -\sqrt{x^2}$, so

$$\lim_{x\to-\infty} \frac{x-2}{\sqrt{x^2+x-2}} = \lim_{x\to-\infty} \frac{1-\frac{1}{x}}{-\sqrt{1+\frac{1}{x}-\frac{2}{x^2}}} = \frac{1-0}{-\sqrt{1+0-0}} = -1$$

2. $f''(x) = 4x^3 - 12x^2 = 4x^2(x-3)$, which is 0 when $x = 0, 3$ and never undefined; so we consider the intervals $(-\infty, 0)$, $(0,3)$ and $(3,\infty)$. Now, $f''(-1) < 0$, $f''(1) < 0$, and $f''(4) > 0$, so the concavities of f on these intervals are down, down and up respectively. Thus, $x = 3$ is a point of inflection; but x = 0 is not, because the concavity does not change there.

3. If a denotes the constant acceleration in mi/hr^2, then the velocity in mi/hr at time t hours after passing the police car is $v(t) = at + 24$ (because $v(0) = 24$), and so the distance in miles past the police car at that time is $s(t) = \frac{1}{2}at^2 + 24t$ (because $s(0) = 0$). Now $s(2/3) = 20$, so we have $20 = \frac{1}{2}a(2/3)^2 + 24(2/3) = (2/9)a + 16$, so $a = (9/2)(20-16) = 18$ mi/hr^2.

4. With $Q = (x,y) = (x, 2 - \frac{1}{2}x)$, the rectangle's area is $A = xy = x(2 - \frac{1}{2}x) = 2x - \frac{1}{2}x^2$, so $dA/dx = 2 - x$, which is never undefined and equal to 0 when $x = 2$. It is clear that the minimum area of 0 square units is attained when $x = 0$ and $x = 4$, when Q is on the boundaries of the first quadrant (at $(0,2)$ and $(4,0)$ respectively), and area is a maximum when $x = 2$, at which point the area is $4 - 2 = 2$ square units. The height is $PQ = OR = y = 2 - \frac{1}{2}(2) = 1$ at this point, so the rectangle giving the maximum area is not a square.

5. Let x denote the horizontal distance between the child and the point on the ground directly below the kite, and let z denote the length of string deployed (i.e., the diagonal distance from child to kite), both measured in feet. Then $z^2 = x^2 + 40^2$; so, first, when $z = 50$, $x = \sqrt{50^2 - 40^2} = 30$, and second, $2z(dz/dt) = 2x(dx/dt)$. Now $dx/dt = 0.5$ ft/sec, so at the time when $z = 50$, we have $dz/dt = [2(30)(.5)]/[2(50)j = 0.3$ ft/sec.

6. (a)

$$\frac{1}{2-(-1)} \int_{-1}^{2} (3x^2+5)dx = \frac{1}{3}\left[x^3+5x\right]_{-1}^{2} = \frac{1}{3}\left((8-(-1)) + 5(2-(-1))\right) = 8 \ .$$

   (b) Let $u = 1 + \sin x$; then $du = \cos x\, dx$, and as x varies from 0 to $\pi/2$, u varies from 1 to 2. Thus, the given integral can be written

$$\int_1^2 \frac{du}{u} = [\ln u]_1^2 = \ln 2 \ .$$

   (c) $[\arctan x]_0^{\sqrt{3}} = \arctan(\sqrt{3}) - \arctan(0) = \pi/3$.
   (d) Because the integrand is an odd function and the interval is symmetric about the origin, the value of the integral is 0.

7. $(1+2+3+2)\cdot 2(\frac{1}{4}) = 4$ pounds.

8. $F(x) = \int_1^x \sqrt{t^3+4}\, dt$

9. $D(\ln(-x)) = (1/(-x))(-1) = 1/x$. Thus, we have $\int(1/x)dx = \ln|x| + C$ for all $x \neq 0$.