A *differential equation* (in x and y) assumes y is a function of x; it is an equation that relates x, y and $y' = dy/dx$ (and maybe $y'', y''', \ldots$). A *solution* for the DE is a function $y = f(x)$ that, when substituted for y, makes a statement that is true for all values of x. The *order* is the number of the highest derivative of y that appears in the DE.

(For a first-order DE, the family of all solutions is usually a family of functions, distinguished by the value of one constant, or "parameter".)

An *initial value problem* is a differential equation in x and y and a specified value for y at a given value of x.

(Substituting in the values of x and y gives the value of the parameter, so that we know which function in the family of solutions to the DE is the solution to the IVP.)

Ex: Which of these are solutions of $y' = x + xy$? Do any of these satisfy $y(0) = 3$?

(a) $y = x + 1$

(b) $y = e^x$

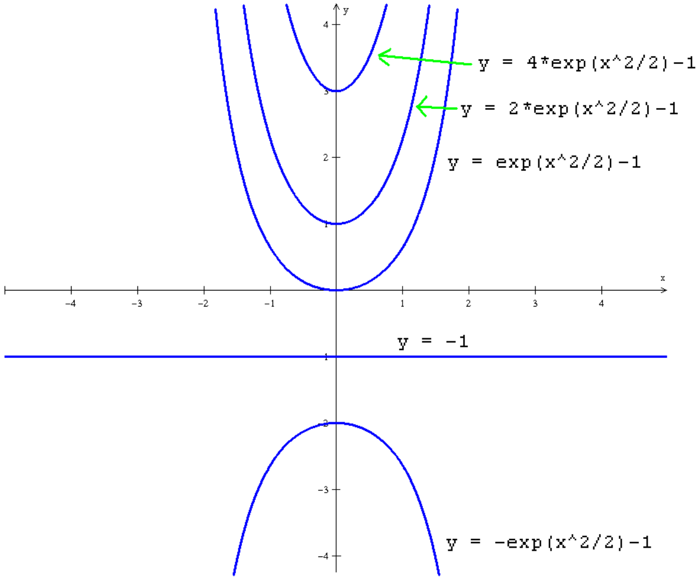
(c) $y = e^{x^2/2}$

(d) $y = 4e^{x^2/2} - 1$

(e) $y = 2e^{x^2/2} - 1$

Where did these come from? Well, $y' = x + xy$ is "separable":

$$\begin{aligned}
\frac{dy}{dx} &= x(1+y) \\
\frac{dy}{1+y} &= x\,dx \\
\int \frac{dy}{1+y} &= \int x\,dx \\
\ln|1+y| &= \frac{1}{2}x^2 + C \\
1+y &= \pm e^{x^2/2} e^C \\
y &= Ke^{x^2/2} - 1
\end{aligned}$$


y = 4*exp(x^2/2)-1
y = 2*exp(x^2/2)-1
y = exp(x^2/2)-1
y = -1
y = -exp(x^2/2)-1
x
y
-4
-3
-2
-1
1
2
3
4

So which of this family of curves satisfies the initial condition $y(0) = 3$? Use the initial condition to determine the value of the parameter K:

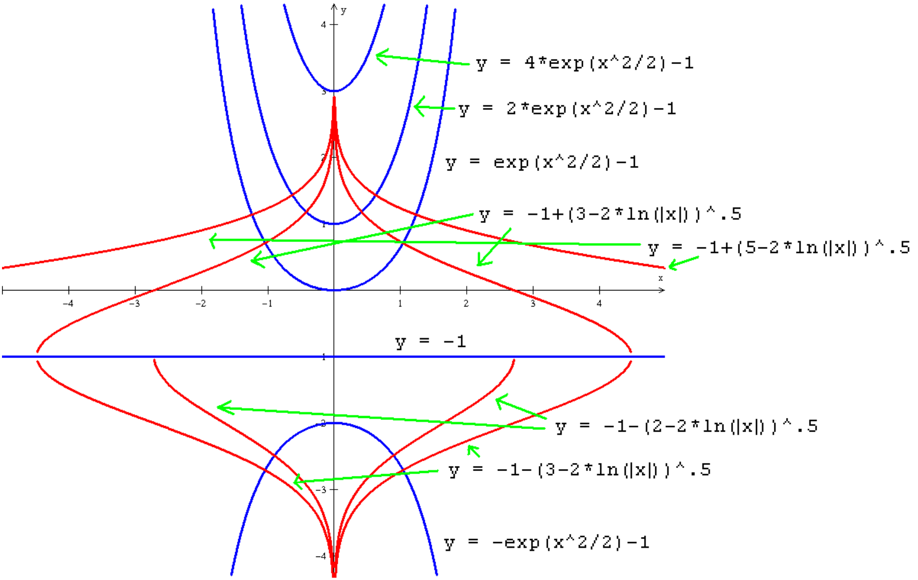
$$
\begin{aligned}
3 &= Ke^{0^2/2} - 1 \\
4 &= K
\end{aligned}
$$

So the solution to the initial value problem is $y = 4e^{x^2/2} - 1$.

Two sets of one-parameter families of functions are *orthogonal trajectories* if, wherever a function in one set intersects one of the functions in the other set, they are perpendicular to each other, in other words, when their slopes, i.e., their derivatives, are negative reciprocals.

In the family of solutions to the earlier example, because the slopes were given by $y' = x(1+y)$, the trajectories that are orthogonal to these are the solutions to $y' = -1/(x(1+y))$. This one is separable, too:

$$\begin{aligned} \int (1+y)\,dy &= \int -\frac{dx}{x} \\ y + \frac{1}{2}y^2 &= -\ln|x| + C \\ y^2 + 2y + (2\ln|x| - 2C) &= 0 \\ y = \frac{-2 \pm \sqrt{4 - 4(2\ln|x| - 2C)}}{2} &= -1 \pm \sqrt{K - 2\ln|x|} \end{aligned}$$


y = 4*exp(x^2/2)-1
y = 2*exp(x^2/2)-1
y = exp(x^2/2)-1
y = -1+(3-2*ln(|x|))^.5
y = -1+(5-2*ln(|x|))^.5
y = -1
y = -1-(2-2*ln(|x|))^.5
y = -1-(3-2*ln(|x|))^.5
y = -exp(x^2/2)-1

And just for practice, which one of these curves satisfies $y(0) = 3$?

$$3 = -1 \pm \sqrt{K - 2 \ln |0|}$$

But 0 has no natural log, so there is no solution to the initial value problem

$$y' = \frac{-1}{x + xy} \ , \qquad y(0) = 3 \ .$$