Integrals by Partial Fractions

Problem: To find the antiderivative of a rational function (a polynomial over a polynomial) $p(x)/r(x)$:

Step 1: Arrange $\deg(r(x)) > \deg(p(x))$ (by long division, if necessary) and the leading coefficient of $r(x)$ is 1 (by taking a constant through the integral, if necessary).

Step 2: Factor

$$r(x) = \ell_1(x)^{e_1}\ell_2(x)^{e_2}\ldots\ell_m(x)^{e_m}q_1(x)^{f_1}q_2(x)^{f_2}\ldots q_n(x)^{f_n}\ ,$$

where the $\ell_i(x) = x - a_i$ are distinct linear factors and the $q_j(x) = (x - b_j)^2 + c_j$ are distinct irreducible quadratic factors (i.e., each c_j is positive).

Step 3: Write

$$\frac{p(x)}{r(x)} = \sum_{i=1}^{m} \sum_{g=1}^{e_i} \frac{A_{ig}}{(x-a_i)^g} + \sum_{j=1}^{n} \sum_{h=1}^{f_j} \frac{B_{jh}(x-b_j)+C_{jh}}{((x-b_j)^2+c_j)^h} \ ,$$

where the capital letters are constants are to be determined.

Step 4: Multiply both sides by $r(x)$ to get two equal polynomials, $p(x)$ on the left and a mess on the right.

Step 5: Get a system of $\deg(r(x))$ linear equations in the constants A_{ig}, B_{jh}, C_{jh} by either:

(a) setting equal the coefficients of each power of x on both sides of the equation; or

(b) picking $\deg(r(x))$ values for x and substituting into both sides.

Step 6: Solve the system to find the values of A_{ig}, B_{jh}, C_{jh}.

Step 7: Integrate all terms (and add $+C$ at the end):

(a)

$$\int \frac{A_{i1}\,dx}{x-a_i} = A_{i1}\ln|x-a_i|$$

(b) For $g>1$,

$$\int \frac{A_{ig}\,dx}{(x-a_i)^g} = -\frac{A_{ig}}{g-1}(x-a_1)^{1-g}$$

(c)

$$\int \frac{B_{j1}(x-b_j)+C_{j1}}{(x-b_j)^2+c_j}\,dx$$
$$= \frac{B_{j1}}{2}\ln((x-b_j)^2+c_j) + \frac{C_{j1}}{\sqrt{c_j}}\tan^{-1}\frac{x-b_j}{\sqrt{c_j}}$$

(d) For $h > 1$,

$$\int \frac{B_{jh}(x - b_j) + C_{jh}}{((x - b_j)^2 + c_j)^h}\, dx$$

$$= -\frac{B_{jh}}{2(h-1)}((x - b_j)^2 + c_j)^{1-h}$$

$$+ C_{jh} c_j^{-h+(1/2)} \int \cos^{2h-2}\theta \, d\theta$$

where

$$x - b_j = \sqrt{c_j} \tan\theta$$

$$dx = \sqrt{c_j} \sec^2\theta \, d\theta$$

$$(x - b_j)^2 + c_j = c_j \sec^2\theta$$