$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$$

$$[\quad u = \cos x \implies du = -\sin x \, dx \quad]$$

$$= -\int \frac{du}{u} = -\ln|u| + C = \ln\left|\frac{1}{u}\right| + C$$

$$= \ln\left|\frac{1}{\cos x}\right| + C = \ln|\sec x| + C \ .$$

Similarly,

$$\int \cot x \, dx = \ln|\sin x| + C \ .$$

$$\int \sec x \, dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx$$

$$\left[\quad u = \sec x + \tan x \right.$$

$$\left. \implies du = (\sec x \tan x + \sec^2 x) \, dx \quad \right]$$

$$= \int \frac{du}{u} = \ln |u| + C = \ln |\sec x + \tan x| + C \ .$$

Similarly,

$$\int \csc x \, dx = \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x} \, dx$$

$$\left[\quad u = \csc x + \cot x \right.$$

$$\left. \implies du = (-\csc x \cot x - \csc^2 x) \, dx \quad \right]$$

$$= -\int \frac{du}{u} = -\ln|u| + C = -\ln|\csc x + \cot x| + C$$

$$= \ln\left|\frac{1}{\csc x + \cot x}\right| + C = \ln\left|\frac{\csc x - \cot x}{\csc^2 x - \cot^2 x}\right| + C$$

$$= \ln|\csc x - \cot x| + C$$

because $\csc^2 x - \cot^2 x = 1$.

$$\int \csc^3 x\,dx = \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x}\,dx$$

$$\left[\quad u = \csc x\ , \quad dv = \csc^2 x\,dx \right.$$

$$\left. \implies du = -\csc x \cot x\,dx\ , \quad v = -\cot x \quad \right]$$

$$= -\csc x \cot x - \int \csc x \cot^2 x\,dx$$

$$= -\csc x \cot x - \int \csc x(\csc^2 x - 1)\,dx$$

$$= -\csc x \cot x - \int \csc^3 x\,dx + \int \csc x\,dx$$

$$2\int \csc^3 x\,dx = -\csc x \cot x + \int \csc x\,dx$$

$$= -\csc x \cot x + \ln|\csc x - \cot x| + C_0$$

$$\int \csc^3 x\,dx = -\frac{1}{2}\csc x \cot x + \frac{1}{2}\ln|\csc x - \cot x| + C\ .$$