

$$\begin{aligned}
 \int \tan x \, dx &= \int \frac{\sin x}{\cos x} \, dx \\
 &\quad [\quad u = \cos x \implies du = -\sin x \, dx \quad] \\
 &= - \int \frac{du}{u} = -\ln|u| + C = \ln\left|\frac{1}{u}\right| + C \\
 &= \ln\left|\frac{1}{\cos x}\right| + C = \ln|\sec x| + C .
 \end{aligned}$$

Similarly,

$$\int \cot x \, dx = \ln|\sin x| + C .$$

$$\begin{aligned}\int \sec x \, dx &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx \\&\quad [\quad u = \sec x + \tan x \\&\quad \implies du = (\sec x \tan x + \sec^2 x) \, dx \quad] \\&= \int \frac{du}{u} = \ln |u| + C = \ln |\sec x + \tan x| + C .\end{aligned}$$

Similarly,

$$\begin{aligned}\int \csc x \, dx &= \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x} \, dx \\&\quad [\quad u = \csc x + \cot x \\&\qquad \Rightarrow du = (-\csc x \cot x - \csc^2 x) \, dx \quad] \\&= - \int \frac{du}{u} = -\ln|u| + C = -\ln|\csc x + \cot x| + C \\&= \ln \left| \frac{1}{\csc x + \cot x} \right| + C = \ln \left| \frac{\csc x - \cot x}{\csc^2 x - \cot^2 x} \right| + C \\&= \ln |\csc x - \cot x| + C\end{aligned}$$

because $\csc^2 x - \cot^2 x = 1$.

$$\int \csc^3 x \, dx = \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x} \, dx$$

[$u = \csc x$, $dv = \csc^2 x \, dx$
 $\Rightarrow du = -\csc x \cot x \, dx$, $v = -\cot x$]

$$\begin{aligned}&= -\csc x \cot x - \int \csc x \cot^2 x \, dx \\&= -\csc x \cot x - \int \csc x (\csc^2 x - 1) \, dx \\&= -\csc x \cot x - \int \csc^3 x \, dx + \int \csc x \, dx\end{aligned}$$

$$\begin{aligned}2 \int \csc^3 x \, dx &= -\csc x \cot x + \int \csc x \, dx \\&= -\csc x \cot x + \ln |\csc x - \cot x| + C_0\end{aligned}$$

$$\int \csc^3 x \, dx = -\frac{1}{2} \csc x \cot x + \frac{1}{2} \ln |\csc x - \cot x| + C .$$