28 September 2010

Math 112 — Exam I

Show all work clearly; an answer with no justifying computations may not receive credit (except in the "set up but do not evaluate" problems).

1. *(11 points)* Find the following derivatives, and simplify if possible:

$$\text{(a)}\ \frac{d}{dx}\left(\tan^{-1}\frac{x}{4}\right) \qquad \text{(b)}\ \frac{d}{dx}\left(\sin^{-1}\frac{x-3}{3}\right)$$

   (I know "simplify" is a subjective term, but do what you can. These derivatives reappear in the next few sections of the course.)

2. *(21 points)* Find the following limits:

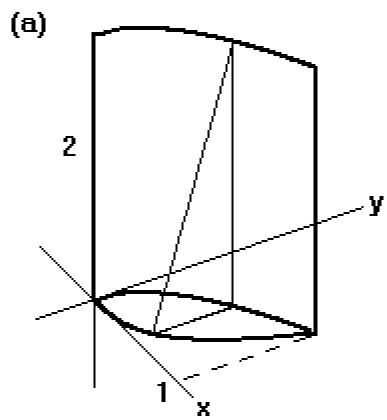
$$\text{(a)}\ \lim_{x\to\infty}(x-\sqrt{x^2-1}) \qquad \text{(b)}\ \lim_{x\to 2}\frac{x^2-2x+3}{x^2+3} \qquad \text{(c)}\ \lim_{x\to 1^+} x^{1/(x-1)}$$

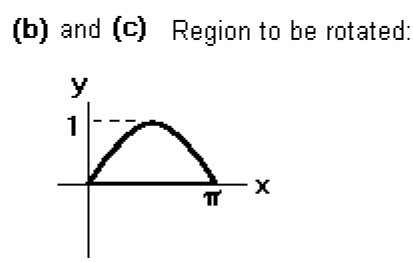
3. *(18 points)* Set up but do not evaluate a definite integral that represents the volume of each of the following solids:

   (a) The base is the region bounded by the curves $y = x^3$ and $x = y^2$, and the cross sections perpendicular to the x-axis are right triangles with height 2.

   (b) The solid of rotation generated by rotating about the x-axis the region above the x-axis and under $y = \sin x$ between $x = 0$ and $x = \pi$.

   (c) The solid of rotation generated by rotating the same region as in (c) about the line $x = -1$.





4. *(24 points)*

   (a) Find the antiderivative $\int \sin^2 x\,dx$, not by a cosine double-angle formula, but by integration by parts (and a simpler trig identity).

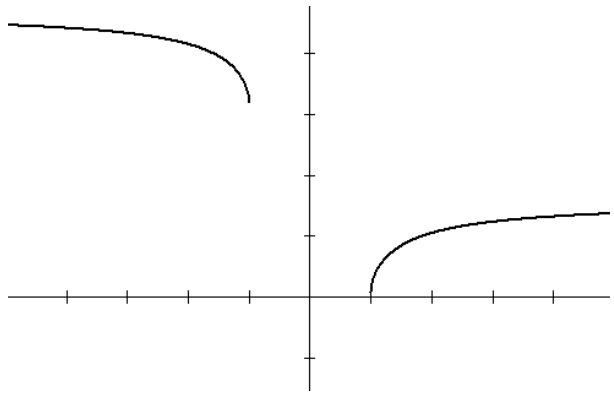
   (b) If you *had* used a cosine double-angle formula, the result would have been $(x/2) - (1/4)\sin 2x + C$. Does this result agree with yours in (a)? Explain.

   (c) Using the antiderivative in (b), find the average value of $\sin^2 x$ on the x-interval $[0, \pi/4]$.

5. *(16 points)* Evaluate the following integrals:

$$\text{(a)} \int \sin^3 x \cos^4 x \, dx \qquad\qquad \text{(b)} \int \cos(\sin x) \cos x \, dx$$

6. *(10 points)* Letting the inverse secant take its values in the intervals $[0, \pi/2)$ and $[\pi, 3\pi/2)$, explain why the derivative of inverse secant is what it is. (You will probably want to start with $y = \sec^{-1} x$, solve for x, differentiate, and solve for dy/dx.) Recall that, with this choice of range, the graph of the inverse secant is



Some possibly useful formulas:

$$\begin{aligned}
\sin 2A &= 2 \sin A \cos A \\
\cos 2A &= \cos^2 A - \sin^2 A \\
&= 2\cos^2 A - 1 \\
&= 1 - 2\sin^2 A \\
1 &= \sin^2 A + \cos^2 A
\end{aligned}$$

$$V = \int_a^b (\text{area of cross section})\, d(\text{thickness})$$

$$V = 2\pi \int_a^b (\text{radius})(\text{height})\, d(\text{thickness})$$

$$y - y_0 = m(x - x_0)$$

Solutions to Exam I:

1.

(a) $$\frac{d}{dx}\left(\tan^{-1}\frac{x}{4}\right) = \frac{1}{\left(\frac{x}{4}\right)^2+1}\cdot\frac{1}{4} = \frac{1}{4\left(\frac{x^2}{16}\right)+4} = \frac{4}{x^2+16}$$

(b) $$\frac{d}{dx}\left(\sin^{-1}\frac{x-3}{3}\right) = \frac{1}{\sqrt{1-\left(\frac{x-3}{3}\right)^2}}\cdot\frac{1}{3} = \frac{1}{\sqrt{9-(x-3)^2}}$$
$$= \frac{1}{\sqrt{9-(x^2-6x+9)}} = \frac{1}{\sqrt{6x-x^2}}$$

2. (a) The indeterminate difference means we want to rewrite it as a quotient:

$$\lim_{x\to infty}(x-\sqrt{x^2-1}) = \lim_{x\to\infty}\frac{x^2-(x^2-1)}{x+\sqrt{x^2-1}} = \lim_{x\to\infty}\frac{1}{x+\sqrt{x^2-1}} = 0\ .$$

We didn't even need L'Hôpital's Rule.

(b) The denominator does not approach 0 as $x \to 2$, so l'Hospital's rule does not apply, but we can evaluate by substitution:

$$\lim_{x\to 2}\frac{x^2-2x+3}{x^2+3} = \frac{2^2-2(2)+3}{2^2+3} = \frac{3}{7}\ .$$

(c) Because the form 1^∞ is an exponential indeterminate form, we find the limit of the log of the function:

$$\lim_{x\to 1}\ln(x^{1/(x-1)}) = \lim_{x\to 1}\frac{\ln x}{x-1} \overset{\text{H}}{=} \lim_{x\to 1}\frac{1/x}{1} = 1\ ,$$

so the desired limit is $e^1 = e$.

3. In each case, we slice perpendicular to the x-axis, so the small piece of thickness is dx:

(a) $\displaystyle\int_0^1 \frac{1}{2}(\sqrt{x}-x^3)(2)\,dx$ (b) $\displaystyle\int_0^\pi \pi(\sin x)^2\,dx$ (c) $\displaystyle 2\pi\int_0^\pi (x+1)\sin x\,dx$

4. (a) Set $u = \sin x$ and $dv = \sin x\,dx$; then $du = \cos x$ and $v = -\cos x$; so

$$\int \sin^2 x\,dx = -\sin x\cos x + \int \cos^2 x\,dx = -\sin x\cos x + \int (1-\sin^2 x)\,dx$$
$$= -\sin x\cos x + x - \int \sin^2 x\,dx$$
$$2\int \sin^2 x\,dx = -\sin x\cos x + x + C$$
$$\int \sin^2 x\,dx = -\frac{1}{2}\sin x\cos x + \frac{x}{2} + C'$$

(b) Yes, it does, because $\sin 2x = 2\sin x\cos x$ — see the "possibly useful formulas". The C' in (a) is even equal to the C in (b).

(c) The average value is

$$\frac{1}{(\pi/4)-0}\int_0^{\pi/4}\sin^2 x\,dx = \frac{4}{\pi}\left[\frac{x}{2}-\frac{1}{4}\sin 2x\right]_0^{\pi/4} = \frac{4}{\pi}\left(\frac{\pi}{8}-\frac{1}{4}\sin\left(\frac{\pi}{2}\right)-0\right)$$
$$= \frac{4}{\pi}\left(\frac{\pi}{8}-\frac{1}{4}\right) = \frac{1}{2}-\frac{1}{\pi}\ .$$

5. (a) Use $\sin^2 x = 1-\cos^2 x$ and set $u=\cos x$, so that $du=-\sin x\,dx$:

$$\int \sin^3 x\cos^4 x\,dx = \int(1-\cos^2 x)\cos^4 x\sin x\,dx = -\int(u^4-u^6)\,du$$
$$= -\frac{1}{5}u^5+\frac{1}{7}u^7+C = -\frac{1}{5}\cos^5 x+\frac{1}{7}\cos^7 x+C\ .$$

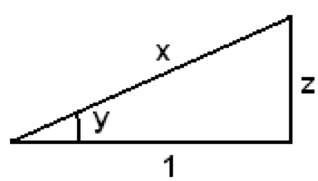
(b) Let $u=\sin x$, so that $du=\cos x\,dx$:

$$\int\cos(\sin x)\cos x\,dx = \int\cos u\,du = \sin u + C = \sin(\sin x)+C\ .$$

6. As the problem suggests,

$$y = \sec^{-1} x$$
$$x = \sec y$$
$$1 = \sec y\tan y\,\frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{1}{\sec y\tan y}$$

Now looking at the triangle



(or using the formula $\tan^2 y+1=\sec^2 y$), we see that $\sec y = x$ gives $\tan y = z/1 = \sqrt{x^2-1}$, the last step by the Pythagorean Theorem. The triangle is valid when y is in $[0,\pi/2)$, so the formula is $dy/dx = 1/(x\sqrt{x^2-1})$ at least there. When y is in $[\pi,3\pi/2)$, $\sec y$ is negative and $\tan y$ is positive, and the graph has a negative slope in this region, so the formula also works in this interval:

$$\frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}}\ .$$