19 October 2010

**Math 112 — Exam II**

Show all work clearly; an answer with no justifying computations may not receive credit (except in the "set up but do not evaluate" problems).

1. *(20 points)* Find the antiderivatives:

$$\text{(a)} \int \sec^3 x \tan x \, dx \qquad\qquad \text{(b)} \int \csc^2 x \cot^2 x \, dx$$

2. *(20 points)* Find the antiderivatives:

$$\text{(a)} \int \frac{x^3\, dx}{\sqrt{9 - x^2}} \qquad\qquad \text{(b)} \int_0^1 \frac{(2x+3)\, dx}{\sqrt{x^2 + 4x + 29}}$$

3. *(10 points)* Write the <u>form</u> of the partial fraction expansion (over the real numbers) of the following rational functions. <u>Do not solve for the constant coefficients!</u>

$$\text{(a)} \ \frac{5x^2 + 3x - 2}{x^4 + 2x^2} \qquad\qquad \text{(b)} \ \frac{x^3 + 2x}{x^2 + 4x + 3}$$

4. *(20 points)* Find the integrals. Note that 1 is a root of the denominator in (a):

$$\text{(a)} \int \frac{x^2\, dx}{x^3 - 1} \qquad\qquad \text{(b)} \int_0^1 \frac{dx}{1 + \sqrt[3]{x}}$$

5. *(5 points)* Rewrite the following as the antiderivative of a rational function (in a variable other than $x$) — then <u>stop</u>; don't go on to evaluate it.

$$\int \frac{dx}{3 - 5\sin x} \ .$$

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Some possibly useful formulas:

If $t = \tan(x/2)$, then

$$\cos x = \frac{1 - t^2}{1 + t^2} \qquad \sin x = \frac{2t}{1 + t^2} \qquad dx = \frac{2\, dt}{1 + t^2}$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \frac{1}{2}(1 + \cos 2x) \qquad \sin^2 x + \cos^2 x = 1$$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C \qquad \int \tan x \, dx = \ln|\sec x| + C$$

$$\int \frac{du}{u^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C \qquad \int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a} + C$$

**Solutions to Exam II:**

1. In (a), let $u = \sec x$; then $du = \sec x \tan x\, dx$. In (b), let $u = \cot x$; then $du = -\csc^2 x\, dx$:

$$\text{(a)} \int \sec^3 x \tan x\, dx = \int u^2\, du = \frac{1}{3}u^3 + C = \frac{1}{3}\sec^3 x + C$$
$$\text{(b)} \int \csc^2 x \cot x\, dx = -\int u^2\, du = -\frac{1}{3}u^3 + C = -\frac{1}{3}\cot^3 x + C$$

2. (a) Let $x = 3\sin\theta$; then $dx = 3\cos\theta\, d\theta$ and $\sqrt{9-x^2} = 3\cos\theta$, so the integral becomes

$$\begin{aligned}
\int \frac{(27\sin^3\theta)(3\cos\theta\, d\theta)}{3\cos\theta} &= 27\int \sin^3\theta\, d\theta = 27\int (1-\cos^2\theta)\sin\theta\, d\theta \\
&= -27\int (1-u^2)\, du = -27(u - \frac{1}{3}u^3) + C \\
&= -27\cos\theta + 9\cos^3\theta + C \\
&= -9\sqrt{9-x^2} + \frac{1}{3}(9-x^2)^{3/2} + C
\end{aligned}$$

where we have finished using the substitution $u = \cos\theta$, so that $du = -\sin\theta$.

(b)

$$\begin{aligned}
\int_0^1 \frac{(2x+3)\, dx}{\sqrt{x^2+4x+29}} &= \int_0^1 \frac{(2x+3)\, dx}{\sqrt{(x^2+4x+4)+29-4}} = \int_0^1 \frac{(2(x+2)+3-4)\, dx}{\sqrt{(x+2)^2+25}} \\
&= \int_2^3 \frac{(2u-1)\, du}{\sqrt{u^2+25}} \qquad \text{where } u = x+2 \\
&= \int_2^3 \left( \frac{2u}{\sqrt{u^2+25}} - \frac{1}{\sqrt{u^2+25}} \right) du
\end{aligned}$$

Each of these could be attacked with the substitution $u = 5\tan\theta$, so that $du = 5\sec^2\theta\, d\theta$ and $\sqrt{u^2+25} = 5\sec\theta$, and as $u$ varies from 2 to 3, $\theta$ varies from $\arctan(2/5)$ to $\arctan(3/5)$:

$$\begin{aligned}
&= \int_{\arctan(2/5)}^{\arctan(3/5)} \left( \frac{10\tan\theta}{5\sec\theta} - \frac{1}{5\sec\theta} \right) (5\sec^2\theta)\, d\theta \\
&= \int_{\arctan(2/5)}^{\arctan(3/5)} (10\tan\theta\sec\theta - \sec\theta)\, d\theta \\
&= [10\sec\theta - \ln|\sec\theta + \tan\theta|]_{\arctan(2/5)}^{\arctan(3/5)} \\
&= 10\left( \frac{\sqrt{34}}{5} - \frac{\sqrt{29}}{5} \right) - \left( \ln\left( \frac{\sqrt{34}}{5} + \frac{3}{5} \right) - \ln\left( \frac{\sqrt{29}}{5} + \frac{2}{5} \right) \right) \\
&= 2\left( \sqrt{34} - \sqrt{29} \right) - \left( \ln\left( \frac{\sqrt{34}}{5} + \frac{3}{5} \right) - \ln\left( \frac{\sqrt{29}}{5} + \frac{2}{5} \right) \right),
\end{aligned}$$

because the secant of the angle whose tangent is $3/5$ is $\sqrt{34}/5$ (draw the triangle) and similarly for $2/5$. This number is approximately 0.713, but who cares?

3. (a)
$$\frac{5x^2+3-2}{x^4+2x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+2} \ .$$

(b) Begin with a long division: $x^3+2x = (x^2+4x+3)(x-4)+15x-12$. Also, $x^2+4x+3 = (x+3)(x+1)$:
$$\frac{x^3+2x}{x^2+4x+3} = x-4+\frac{A}{x+3}+\frac{B}{x+1} \ .$$

4. (a) The substitution $u = x^3-1$ gives $du = 3x^2\,dx$, so the easy way to solve this one is
$$\int \frac{x^2\,dx}{x^3-1} = \frac{1}{3}\int \frac{du}{u} = \frac{1}{3}\ln|u|+C = \frac{1}{3}\ln|x^3-1|+C \ .$$
A question I <u>should</u> have asked was
$$\int \frac{x^2+2}{x^3+1}\,dx \ :$$
We can factor $x^3-1 = (x-1)(x^2+x+1)$ and the second factor is irreducible. That factor can be written as $(x+(1/2))^2+(3/4)$, so we need to write
$$\begin{aligned}
\frac{x^2}{x^3-1} &= \frac{A}{x-1} + \frac{B(x+1/2)+C}{(x+(1/2))^2+(3/4)} \\
x^2+2 &= A(x^2+x+1)+(B(x+1/2)+C)(x-1) \\
&= A(x^2+x+1)+(Bx+(1/2)B+C)(x-1) \\
&= (A+B)x^2+(A-B+(1/2)B+C)x+(A-(1/2)B-C)
\end{aligned}$$

$$\begin{array}{ccccccc}
A & + & B & & & = & 1 \\
A & - & (1/2)B & + & C & = & 0 \\
A & - & (1/2)B & - & C & = & 2
\end{array}
\quad \Longrightarrow \quad
\begin{array}{ccc}
A & = & 1 \\
B & = & 0 \\
C & = & -1
\end{array}$$

So
$$\begin{aligned}
\int \frac{x^2+2}{x^3-1}\,dx &= \int \left(\frac{1}{x-1} - \frac{1}{(x+1/2)^2+3/4}\right)dx \\
&= \ln|x-1| - \frac{2}{\sqrt{3}}\arctan\left(\frac{x+1/2}{\sqrt{3}/2}\right)+C \ .
\end{aligned}$$

(b) Let $u = \sqrt[3]{x}$; then $x = u^3$, so $dx = 3u^2\,du$ and $1+\sqrt[3]{x} = 1+u$, and as $x$ goes from 0 to 1, so does $u$: Dividing $u+1$ int $3u^2$ gives a quotient of $3u-3$ and a remainder of 3, so
$$\begin{aligned}
\int_0^1 \frac{dx}{1+\sqrt[3]{x}} &= \int_0^1 \frac{3u^2\,du}{1-u} = \int_0^1 \left(3u-3+\frac{3}{u+1}\right)du \\
&= \left[\frac{3}{2}u^2-3u+3\ln|u+1|\right]_0^1 = \frac{3}{2}-3+3\ln 2 = 3\ln 2-\frac{3}{2} \ ,
\end{aligned}$$
which is about 0.58, but who cares?

5. Use the substitution $t = \tan(x/2)$; then
$$\int \frac{dx}{3-5\sin x} = \int \frac{\frac{2\,dt}{1+t^2}}{3-5\frac{2t}{1+t^2}} = \int \frac{2\,dt}{3(1+t^2)-5(2t)} = \int \frac{2\,dt}{3t^2-10t+3} \ .$$