16 November 2010

Math 112 — Exam III

Show all work clearly; an answer with no justifying computations may not receive credit (except for the "write down but do not evaluate" questions). (Possible high score 90 points.)

1. *(30 points)* Let $f(x) = \sqrt{4+x^2}$ for $0 \le x \le 2$. It is desired to rotate the graph about the x-axis and find the area of the resulting surface of revolution.

   (a) Write but <u>do not evaluate</u> an integral that gives the desired surface area.

   (b) Suppose (falsely) that the integral in (a) is

   $$\pi \int_0^2 \sqrt{7+3x^2}\,dx\ .$$

   For this (incorrect) integral, write the Simpson approximation with $n = 4$. (You don't need to evaluate anything; for example, the value of the integrand at $x = 0.3$ could be $\sqrt{7+3(0.3)^2}$, or maybe $\sqrt{7.27}$, but certainly not 2.696.)

   (c) Suppose (falsely) that the fourth derivative of the integrand in (b) is

   $$\sqrt{3+x^2}\ .$$

   On this (false) basis, find a bound for the error in the approximation you found in (b). (Again, you don't have to compute anything; just write an expression for the answer.)

2. *(24 points)* Evaluate each of the following integrals, if possible.

   (a) $\displaystyle\int_4^\infty e^{-x/2}\,dx$ (b) $\displaystyle\int_0^2 \frac{dx}{(x-1)^2}$ (c) $\displaystyle\int_{-\infty}^\infty \frac{dx}{x^2+16}$

3. *(8 points)* Verify that, for any constant value of C, $y = (\ln x + C)/x$ is a solution to the differential equation $x^2y' + xy = 1$.

4. *(16 points)*

   (a) Let $y(x)$ denote the solution to the initial value problem $y' = y + 1$ with $y(0) = 3$. Use Euler's method to approximate the value $y(3)$, with two (large) step sizes: (i) 3 and (ii) 1.

   (b) Comparing your results in (a), do you believe they are underestimates or overestimates? (You might want to consider the concavity of y.)

5. *(12 points)*

   (a) (The table of integrals may be helpful for this one.) Find the family of solutions to the differential equation

   $$xy' = y^2 - 1\ .$$

   (b) Which member of the family in (a) satisfies $y(1) = 2$?

Some possibly useful formulas:

$$K \geq |f''(x)|\ ,\ a \leq x \leq b\ :\quad |E_T| \leq \frac{K(b-a)^3}{12n^2}\ ,\quad |E_M| \leq \frac{K(b-a)^3}{24n^2}$$

$$K \geq |f^{(4)}(x)|\ ,\ a \leq x \leq b\ :\quad |E_S| \leq \frac{K(b-a)^5}{180n^4}$$

$$L = \int_a^b \sqrt{1+(f'(x))^2}\,dx\ ,\qquad A = 2\pi\int_a^b f(x)\sqrt{1+(f'(x))^2}\,dx$$

$$\int_0^1 \frac{1}{x^p}\,dx \text{ converges iff } p<1,\qquad \int_1^\infty \frac{1}{x^p}\,dx \text{ converges iff } p>1$$

Table of Integrals

(numbering as in Stewart)

Basic forms:

1. $\displaystyle\int u\,dv = uv - \int v\,du$

5. $\displaystyle\int a^u\,du = \frac{a^u}{\ln a} + C$

12. $\displaystyle\int \tan u\,du = \ln|\sec u| + C$

13. $\displaystyle\int \cot u\,du = \ln|\sin u| + C$

14. $\displaystyle\int \sec u\,du = \ln|\sec u + \tan u| + C$

15. $\displaystyle\int \csc u\,du = \ln|\csc u - \cot u| + C$

16. $\displaystyle\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + C$

17. $\displaystyle\int \frac{du}{u^2+a^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$

18. $\displaystyle\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{u}{a} + C$

19. $\displaystyle\int \frac{du}{a^2-u^2} = \frac{1}{2a}\ln\left|\frac{u+a}{u-a}\right| + C$

20. $\displaystyle\int \frac{du}{u^2-a^2} = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right| + C$

Forms involving $\sqrt{a^2 \pm u^2}$, $a > 0$:

21. $$\int \sqrt{a^2+u^2}\,du = \frac{u}{2}\sqrt{a^2+u^2} + \frac{a^2}{2}\ln(u+\sqrt{a^2+u^2}) + C$$

22. $$\int u^2\sqrt{a^2+u^2}\,du = \frac{u}{8}(a^2+2u^2)\sqrt{a^2+u^2} - \frac{a^4}{8}\ln(u+\sqrt{a^2+u^2}) + C$$

23. $$\int \frac{\sqrt{a^2+u^2}}{u}\,du = \sqrt{a^2+u^2} - a\ln\left|\frac{a+\sqrt{a^2+u^2}}{u}\right| + C$$

24. $$\int \frac{\sqrt{a^2+u^2}}{u^2}\,du = -\frac{\sqrt{a^2+u^2}}{u} + \ln(u+\sqrt{a^2+u^2}) + C$$

25. $$\int \frac{du}{\sqrt{a^2+u^2}} = \ln(u+\sqrt{a^2+u^2}) + C$$

26. $$\int \frac{u^2\,du}{\sqrt{a^2+u^2}} = \frac{u}{2}\sqrt{a^2+u^2} - \frac{a^2}{2}\ln(u+\sqrt{a^2+u^2}) + C$$

27. $$\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\ln\left|\frac{\sqrt{a^2+u^2}+a}{u}\right| + C$$

28. $$\int \frac{du}{u^2\sqrt{a^2+u^2}} = -\frac{\sqrt{a^2+u^2}}{a^2u} + C$$

29. $$\int \frac{du}{(a^2+u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2+u^2}} + C$$

30. $$\int \sqrt{a^2-u^2}\,du = \frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$$

31. $$\int u^2\sqrt{a^2-u^2}\,du = \frac{u}{8}(2u^2-a^2)\sqrt{a^2-u^2} - \frac{a^4}{8}\sin^{-1}\frac{u}{a} + C$$

32. $$\int \frac{\sqrt{a^2-u^2}}{u}\,du = \sqrt{a^2-u^2} + a\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C$$

33. $$\int \frac{\sqrt{a^2-u^2}}{u^2}\,du = -\frac{1}{u}\sqrt{a^2-u^2} - \sin^{-1}\frac{u}{a} + C$$

34. $$\int \frac{u^2\,du}{\sqrt{a^2-u^2}} = -\frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$$

35. $$\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C$$

36. $$\int \frac{du}{u^2\sqrt{a^2-u^2}} = -\frac{1}{a^2u}\sqrt{a^2-u^2} + C$$

37. $$\int (a^2-u^2)^{3/2}\,du = -\frac{u^2}{8}(2u^2-5a^2)\sqrt{a^2-u^2} + \frac{3a^4}{8}\sin^{-1}\frac{u}{a} + C$$

38. $$\int \frac{du}{(a^2-u^2)^{3/2}} = \frac{u}{a^2(a^2-u^2)} + C$$

Forms involving $\sqrt{u^2 - a^2}$, $a > 0$:

39. $$\int \sqrt{u^2-a^2}\,du = \frac{u}{2}\sqrt{u^2-a^2} - \frac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right| + C$$

40. $$\int u^2\sqrt{u^2-a^2}\,du = \frac{u}{8}(2u^2-a^2)\sqrt{u^2-a^2} - \frac{a^4}{8}\ln\left|u+\sqrt{u^2-a^2}\right| + C$$

41. $$\int \frac{\sqrt{u^2-a^2}}{u}\,du = \sqrt{u^2-a^2} - a\cos^{-1}\frac{a}{|u|} + C$$

42. $$\int \frac{\sqrt{u^2-a^2}}{u^2}\,du = -\frac{\sqrt{u^2-a^2}}{u} + \ln\left|u+\sqrt{u^2-a^2}\right| + C$$

43. $$\int \frac{du}{\sqrt{u^2-a^2}} = \ln\left|u+\sqrt{u^2-a^2}\right| + C$$

44. $$\int \frac{u^2\,du}{\sqrt{u^2-a^2}} = \frac{u}{2}\sqrt{u^2-a^2} + \frac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right| + C$$

45. $$\int \frac{du}{u^2\sqrt{u^2-a^2}} = \frac{\sqrt{u^2-a^2}}{a^2u} + C$$

46. $$\int \frac{du}{(u^2-a^2)^{3/2}} = -\frac{u}{a^2\sqrt{u^2-a^2}} + C$$

Trigonometric forms

63. $\int \sin^2 u\, du = \frac{1}{2}u - \frac{1}{4}\sin 2u + C$

64. $\int \cos^2 u\, du = \frac{1}{2}u + \frac{1}{4}\sin 2u + C$

65. $\int \tan^2 u\, du = \tan u - u + C$

66. $\int \cot^2 u\, du = -\cot u - u + C$

73. $\int \sin^n u\, du = -\frac{1}{n}\sin^{n-1} u \cos u + \frac{n-1}{n}\int \sin^{n-2} u\, du$

74. $\int \cos^n u\, du = \frac{1}{n}\cos^{n-1} u \sin u + \frac{n-1}{n}\int \cos^{n-2} u\, du$

75. $\int \tan^n u\, du = \frac{1}{n-1}\tan^{n-1} u - \int \tan^{n-2} u\, du$

76. $\int \cot^n u\, du = -\frac{1}{n-1}\cot^{n-1} u - \int \cot^{n-2} u\, du$

77. $\int \sec^n u\, du = \frac{1}{n-1}\tan u \sec^{n-2} u + \frac{n-2}{n-1}\int \sec^{n-2} u\, du$

78. $\int \csc^n u\, du = -\frac{1}{n-1}\cot u \csc^{n-2} u + \frac{n-2}{n-1}\int \csc^{n-2} u\, du$

79. $\int \sin au \sin bu\, du = \frac{\sin(a-b)u}{2(a-b)} - \frac{\sin(a+b)u}{2(a+b)} + C$

80. $\int \cos au \cos bu\, du = \frac{\sin(a-b)u}{2(a-b)} + \frac{\sin(a+b)u}{2(a+b)} + C$

81. $\int \sin au \cos bu\, du = -\frac{\cos(a-b)u}{2(a-b)} - \frac{\cos(a+b)u}{2(a+b)} + C$

82. $\int u \sin u\, du = \sin u - u\cos u + C$

83. $\int u \cos u\, du = \cos u + u\sin u + C$

84. $\int u^n \sin u\, du = -u^n \cos u + n\int u^{n-1}\cos u\, du$

85. $\int u^n \cos u\, du = u^n \sin u - n\int u^{n-1}\sin u\, du$

86. $\int \sin^n u \cos^m u\, du$

$$= -\frac{\sin^{n-1} u \cos^{m+1} u}{n+m} + \frac{n-1}{n+m}\int \sin^{n-2} u \cos^m u\, du$$

$$= \frac{\sin^{n+1} u \cos^{m-1} u}{n+m} + \frac{m-1}{n+m}\int \sin^n u \cos^{m-2} u\, du$$

Inverse trigonometric forms

87. $\int \sin^{-1} u\, du = u\sin^{-1} u + \sqrt{1-u^2} + C$

88. $\int \cos^{-1} u\, du = u\cos^{-1} u - \sqrt{1-u^2} + C$

89. $\int \tan^{-1} u\, du = u\tan^{-1} u + \frac{1}{2}\ln(1-u^2) + C$

90. $\int u\sin^{-1} u\, du = \frac{2u^2-1}{4}\sin^{-1} u + \frac{u\sqrt{1-u^2}}{4} + C$

91. $\int u\cos^{-1} u\, du = \frac{2u^2-1}{4}\cos^{-1} u - \frac{u\sqrt{1-u^2}}{4} + C$

92. $\int u\tan^{-1} u\, du = \frac{u^2+1}{2}\tan^{-1} u - \frac{u}{2} + C$

Exponential and logarithmic forms

96. $\int ue^{au}\, du = \frac{1}{a^2}(au-1)e^{au} + C$

97. $\int u^n e^{au} = \frac{1}{a}u^n e^{au} - \frac{n}{a}\int u^{n-1}e^{au}\, du$

98. $\int e^{au}\sin bu\, du = \frac{e^{au}}{a^2+b^2}(a\sin bu - b\cos bu) + C$

99. $\int e^{au}\cos bu\, du = \frac{e^{au}}{a^2+b^2}(a\cos bu + b\sin bu) + C$

100. $\int \ln u\, du = u\ln u - u + C$

101. $\int u^n \ln u\, du = \frac{u^{n+1}}{(n+1)^2}[(n+1)\ln u - 1] + C$

102. $\int \frac{du}{u\ln u} = \ln|\ln u| = +C$

Solutions to Exam III:

1. (a)

$$2\pi\int_0^2 \sqrt{4+x^2}\sqrt{1+\left(\frac{x}{\sqrt{4+x^2}}\right)^2}\,dx = 2\pi\int_0^2 \sqrt{4+x^2}\sqrt{\frac{2x^2+4}{4+x^2}}\,dx$$
$$= 2\pi\int_0^2 \sqrt{2x^2+4}\,dx$$

(b) The intervals have width $(2-0)/4 = 0.5$, so

$$\pi\frac{2-0}{3}\left[\sqrt{7}+4(\sqrt{7+3(.25)})+2(\sqrt{7+3(1)})+4(\sqrt{7+3(2.25)})+\sqrt{7+3(4)}\right]$$

(c) We need to find a bound for the fourth derivative of the integrand, which we are to assume is given. But this function is clearly increasing for $x \geq 0$ (which we could verify by taking a derivative), so it takes its largest value at the right end, where it is $\sqrt{3+2^2} = \sqrt{7}$. Putting that into the formula for the bound on the error in Simpson's Rule, we get

$$|E_S| \leq \frac{\sqrt{7}(2-0)^5}{180(4^4)} \approx 0.002\ .$$

(But of course you don't need to find the last approximation.)

2. (a) $-2e^{-x/2}|_4^\infty = -2(0-e^{-2}) = 2/e^2$

   (b) This function is unbounded at $x = 1$, inside the interval of integration, so we need to break it into two integrals, and if either is divergent, then so is this integral. But one of them is $\int_1^2 dx/(x-1)^2$, equivalent to $\int_0^1 dx/x^2$, which is divergent by the rule given in the "possibly useful formulas." So this integral is divergent. If you didn't notice that the function was unbounded inside the interval and got

$$\int_0^2 \frac{dx}{(x-1)^2} = [-(x-1)^{-1}]_0^2 = -1-1 = -2\ ,$$

   the fact that the integral of this obviously positive function is apparently negative should have suggested that something was wrong.

   (c) It would be wise to break this one into two improper integrals, from $-\infty$ to 0 and from 0 to ∞, to be sure that each is convergent. But they are, and the value is

$$\left[\frac{1}{4}\tan^{-1}\frac{x}{4}\right]_{-\infty}^{\infty} = \frac{1}{4}\left(\frac{\pi}{2}\right) - \frac{1}{4}\left(-\frac{\pi}{2}\right) = \frac{\pi}{4}\ .$$

3.

$$x^2\frac{d}{dx}\left(\frac{\ln x + C}{x}\right) + x\left(\frac{\ln x + C}{x}\right) = x^2\left(\frac{x(1/x)-(\ln x + C)}{x^2}\right) + \ln x + C$$
$$= 1 - \ln x - C + \ln x + C = 1$$

4. (a) (i) $y(3) \approx y(0) + 3(y(0)+1) = 3 + 3(3+1) = 15$

(ii) $y(1) \approx y(0) + 1(y(0)+1) = 7$, so $y(2) \approx y(1) + 1(y(1)+1) \approx 7 + 1(7+1) = 15$, so $y(3) \approx y(2) + 1(y(2)+1) \approx 15 + 1(15+1) = 31$.

(b) Because $y'' = (y+1)' = y' = y+1 > 0$, y is concave up, the straight-line approximations by Euler's method are underestimates.

5. (a)

$$\begin{aligned} \int \frac{dy}{y^2-1} &= \int \frac{dx}{x} \\ \frac{1}{2}\ln\left|\frac{y-1}{y+1}\right| &= \ln|x| + C \qquad [\#20] \\ \frac{y-1}{y+1} &= Kx^2 \end{aligned}$$

This K is $\pm e^C$, so it is not clear that it could be 0. However, when $K = 0$, the equation becomes $y = 1$, and this is a solution to the differential equation.

(b) $(2-1)/(2+1) = K1^2$ gives $K = 1/3$, so $(y-1)/(y+1) = x^2/3$. (Or, with more algebra, $y = (3+x^2)/(3-x^2)$.)