February 17, 2004

Math 112 — Practice Exam I

Show all work clearly; an answer with no justifying computations may not receive credit (except in the "set up but do not evaluate" problems).

1. *(15 points)* (a) Find the equation of the two tangent lines to the graph of $x^2 + y^2 = (2x^2 + 2y^2 - x)^2$ at the points $(0, \pm\frac{1}{2})$.

   (b) Find $\frac{d}{dx}(\arcsin(2x))$.

2. *(21 points)* Find the following limits:

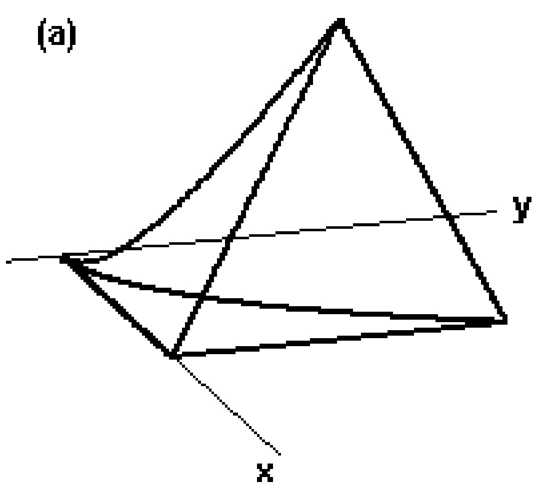
$$\text{(a)} \lim_{x\to\infty}(x - \sqrt{x^2 - 3x}) \qquad \text{(b)} \lim_{x\to 0} x \csc x \qquad \text{(c)} \lim_{x\to\infty}(1 - \frac{1}{x})^x$$

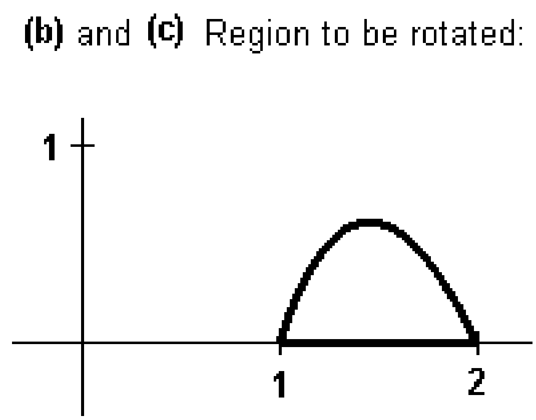
3. *(18 points)* Set up but *do not evaluate* a definite integral that represents the volume of each of the following solids:

   (a) The base is the region bounded by the x-axis, the line $x = 1$ and the parabola $y = x^2$; the cross sections perpendicular to the x-axis are equilateral triangles.

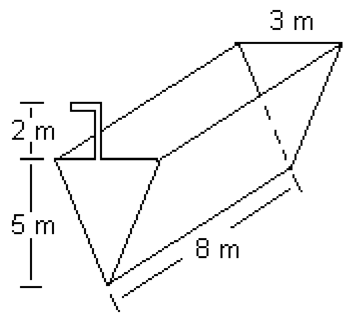
   (b) The solid of rotation generated by rotating about the x-axis the region above the x-axis and under $y = 3(2 - x)\ln x$.

   (c) The solid of rotation generated by rotating the same region as in (c) about the line $x = 3$.





4. *(10 points)* A liquid weighing 9800 N/m^3 fills a tank in the shape below. Set up but *do not evaluate* a definite integral that gives the work done in emptying the tank. Note: The width of the tank is proportional to the distance from the bottom of the tank.



5. *(15 points)* Find the antiderivative $\int (x^2 + 9) \ln x \, dx$ by integration by parts.

6. *(21 points)* Evaluate the following integrals:

(a) $\displaystyle\int \sec^3 x \tan^4 x \, dx$ (b) $\displaystyle\int \cos 2x \sin^4 x \, dx$ (c) $\displaystyle\int \frac{\cos x}{1 + \sin^2 x} \, dx$

Some possibly useful formulas:

$$\begin{aligned}
\sin A \cos B &= \frac{1}{2}(\sin(A+B) + \sin(A-B)) \\
\sin A \sin B &= \frac{1}{2}(\cos(A-B) - \cos(A+B)) \\
\cos A \cos B &= \frac{1}{2}(\cos(A-B) + \cos(A+B))
\end{aligned}$$

$$\begin{aligned}
\sin 2A &= 2 \sin A \cos A \\
\cos 2A &= \cos^2 A - \sin^2 A \\
&= 2\cos^2 A - 1 \\
&= 1 - 2\sin^2 A
\end{aligned}$$

$$\int \csc x \, dx = \ln|\csc x - \cot x| + C \qquad \int \cot x \, dx = \ln|\sin x| + C$$

$$y - y_0 = m(x - x_0)$$

Solutions to Exam I

1. Implicit differentiation of the given equation gives $2x+2yy' = 2(2x^2+2y^2-x)(4x+4yy'-1)$, so dividing by 2 and moving all the terms with a y' to the left side of the equation and all others to the right gives $y'(y-4y(2x^2+2y^2-x)) = (4x-1)(2x^2+2y^2-x)-x$, or

$$y' = \frac{(4x-1)(2x^2+2y^2-x)-x}{y-4y(2x^2+2y^2-x)} \; .$$

At the point $(0, \frac{1}{2})$, we get $y' = [(0-1)(0+\frac{1}{2}-0)]/[\frac{1}{2}-2(0+\frac{1}{2}-0)] = 1$, so the desired tangent line is $y-\frac{1}{2} = 1(x-0)$ (or $y = \frac{1}{2}+x$). At the point $(0,-\frac{1}{2})$, we get $y' = [(0-1)(0+\frac{1}{2}-0)]/[-\frac{1}{2}+2(0+\frac{2}{-}0)] = -1$, so the desired tangent line is $y+\frac{1}{2} = -1(x-0)$ (or $y = -\frac{1}{2}-x$).

(b) $(1/\sqrt{1-(2x)^2})2 = 2/\sqrt{1-4x^2}$.

2. (a) We multiply numerator and denominator (1) by $x+\sqrt{x^2-3x}$ and divide numerator and denominator by x; then the result is clear:

$$\begin{aligned}\lim_{x\to\infty}(x-\sqrt{x^2-3x} &= \lim_{x\to\infty}\frac{x^2-(x^2-3x)}{x+\sqrt{x^2-3x}} = \lim_{x\to\infty}\frac{3x}{x+\sqrt{x^2-3x}} \\ &= \lim_{x\to\infty}\frac{3}{1+\sqrt{1-\frac{3}{x}}} = \frac{3}{1+\sqrt{1+0}} = \frac{3}{2} \; .\end{aligned}$$

Notice that we didn't use l'Hospital's rule; it would be valid (after the first step), but it would not make the limit easier to find.

(b) As $x \to 0$, $\csc x$ approaches $\pm\infty$ (one from the left, the other from the right), so the limit is indeterminate. So we rewrite as a quotient and use l'Hospital's rule (or just recite the limit that was used in finding the derivative of $\sin x$:

$$\lim_{x\to 0} x\csc x = \lim_{x\to 0}\frac{x}{\sin x} \stackrel{H}{=} \lim_{x\to 0}\frac{1}{\cos x} = \frac{1}{1} = 1 \; .$$

(c) Because the form 1^∞ is an exponential indeterminate form, we find the limit of the log of the function:

$$\begin{aligned}\lim_{x\to\infty}\ln\left(\left(1-\frac{1}{x}\right)^x\right) &= \lim_{x\to\infty} x\ln\left(1-\frac{1}{x}\right) = \lim_{x\to\infty}\frac{\ln\left(1-\frac{1}{x}\right)}{x^{-1}} \\ &\stackrel{H}{=} \lim_{x\to\infty}\frac{\left(1-\frac{1}{x}\right)^{-1}(x^{-2})}{-x^{-2}} = \lim_{x\to\infty} -\frac{1}{1-\frac{1}{x}} = -\frac{1}{1-0} = -1 \; ,\end{aligned}$$

so the desired limit is $e^{-1} = 1/e$.

3. For (a), because the height of an equilateral triangle of side s is $(\sqrt{3}/2)s$, its area is $(\sqrt{3}/4)s^2$; and because the slice of the solid at a given x-value is a triangle of side x^2, its area is $(\sqrt{3}/4)x^4$; so

(a) $\displaystyle\int_0^1 \frac{\sqrt{3}}{4}(x^4)dx$ (b) $\displaystyle\int_1^2 \pi(3(2-x)\ln x)^2 dx$ (c) $\displaystyle\int_1^2 2\pi(3-x)(3(2-x)\ln x)dx$

4. A slice of liquid that is x m above the bottom of the tank, of thickness dx, has length 8 m and width w, where $w/x = 3/5$, or $w = (3/5)x$, so its weight in newtons is $9800(8)(3/5)x\,dx$. That slice has to be moved vertically $7 - x$ m. So the total amount of work done is

$$W = \int_0^5 (7-x)9800(8)(3/5)x\,dx\ .$$

5. Because the antiderivative of $\ln x$ is not well-known, let us begin by setting $u = \ln x$ and $dv = (x^2+9)dx$, so that $du = (1/x)dx$ and $v = \frac{1}{3}x^3 + 9x$. Then we get

$$\int (x^2+9)\ln x\,dx = (\frac{1}{3}x^3 + 9x)\ln x - \int \left(\frac{1}{3}x^2 + 9\right)dx = (\frac{1}{3}x^3 + 9x)\ln x - \frac{1}{9}x^3 - 9x + C\ .$$

6. (a) Neither of the obvious substitutions, $u = \tan x$ or $u = \sec x$, seems to work, because the coefficients are inconvenient: using the Pythagorean relation between secant and tangent, we can't conveniently leave two factors of secant or one of tangent. One way to go would be to create a reduction formula for $\int \tan^m x \sec^n x\,dx$, using integration by parts, with $u = \tan^{m-1} x$ and $dv = \sec^n x \tan x\,dx$, so that $du = (m-1)\tan^{m-2} x \sec^2 x\,dx$ and $v = \frac{1}{n}\sec^n x$:

$$\begin{aligned}
\int \tan^m x \sec^n x\,dx &= \frac{1}{n}\tan^{m-1} x \sec^n x - \frac{m-1}{n}\int \tan^{m-2} x \sec^{n+2} x\,dx \\
&= \frac{1}{n}\tan^{m-1} x \sec^n x - \frac{m-1}{n}\int \tan^{m-2} x(\tan^2 x + 1)\sec^n x\,dx \\
&= \frac{1}{n}\tan^{m-1} x \sec^n x - \frac{m-1}{n}\int \tan^{m} x \sec^n x\,dx \\
&\qquad - \frac{m-1}{n}\int \tan^{m-2} x \sec^n x\,dx
\end{aligned}$$

Adding the middle term of the last expression to both ends and dividing both ends by $(n+m-1)/n$, we get

$$\int \tan^m x \sec^n x\,dx = \frac{1}{n+m-1}\tan^{m-1} x \sec^n x - \frac{m-1}{n+m-1}\int \tan^{m-2} x \sec^n x\,dx\ .$$

Using this and the reduction formula on page 481, exercise 48, we get

$$\begin{aligned}
\int \tan^4 x \sec^3 x\,dx &= \frac{1}{6}\tan^3 x \sec^3 x - \frac{1}{2}\int \tan^2 x \sec^3 x\,dx \\
&= \frac{1}{6}\tan^3 x \sec^3 x - \frac{1}{2}\left(\frac{1}{4}\tan x \sec^3 x - \frac{1}{4}\int \sec^3 x\,dx\right) \\
&= \frac{1}{6}\tan^3 x \sec^3 x - \frac{1}{8}\tan x \sec^3 x \\
&\qquad + \frac{1}{8}\left(\frac{1}{2}\tan x \sec x + \frac{1}{2}\int \sec x\,dx\right) \\
&= \frac{1}{6}\tan^3 x \sec^3 x - \frac{1}{8}\tan x \sec^3 x \\
&\qquad + \frac{1}{16}\tan x \sec x + \frac{1}{16}\ln|\sec x + \tan x| + C\ .
\end{aligned}$$

This doesn't use anything we don't already know, but it's complicated. The method suggested in class was to change the integrals into sines and cosines:

$$\int \frac{1}{\cos^3 x}\cdot\frac{\sin^4 x}{\cos^4 x}\,dx = \int \frac{\sin^4 x}{\cos^8 x}\cos x\,dx\ .$$

Now we can replace the $\cos^8 x$ in the denominator with $(1 - \sin^2 x)^4$ and make the substitution $u = \sin x$ (so that $du = \cos x\,dx$): $\int (u^4/(1-u^2)^4)du$. As mentioned in class, this calls for partial fractions, which we won't try to do here. (And no, there won't be one of these on the exam).

(b)

$$\begin{aligned}
\int \cos 2x \sin^4 x\,dx &= \frac{1}{4}\int \cos 2x (1-\cos 2x)^2 dx = \frac{1}{4}\int (\cos 2x - 2\cos^2 2x + \cos^3 2x)dx \\
&= \frac{1}{4}\left(\frac{1}{2}\sin 2x - \int (1+\cos 4x)dx + \int (1-\sin^2 2x)\cos 2x\,dx\right) \\
&\qquad [\text{In the last integral} : u = \sin 2x,\ du = 2\cos 2x\,dx] \\
&= \frac{1}{8}\sin 2x - \frac{1}{4}x - \frac{1}{16}\sin 4x + \frac{1}{8}(\sin 2x - \frac{1}{3}\sin^3 2x) + C\ .
\end{aligned}$$

(c) Let $u = \sin x$, so that $du = \cos x\,dx$. Then

$$\int \frac{\cos x}{1+\sin^2 x}dx = \int \frac{1}{1+u^2}du = \arctan u + C = \arctan(\sin x) + C\ .$$