March 8, 2004

Math 112 — Practice Exam II

Show all work clearly; an answer with no justifying computations may not receive credit (except for the "write down but do not evaluate" questions). If any divergent improper integrals appear, state that they are divergent.

1. *(8 points)* Write down, but *do not evaluate,* an integral that represents the length of the curve $y = \arctan(2x)$ on $0 \le x \le \sqrt{3}/2$.

2. *(10 points)* Write out the *form* of the partial fraction expansion of the rational function

$$\frac{x^6 + 3x^5 - x^3 + 5x + 2}{(x-1)^2(x^2+3)^2}\ .$$

   You need not find the constants in the numerators, but be sure to complete the first step.

3. *(10 points)* Find the centroid of the half-disc $x^2 + y^2 \le a^2$, $y \ge 0$, in terms of the constant a.

4. *(7 points)* Recall that the Laplace transform of a function $f(t)$ (that is continuous for $t \ge 0$) is given by $L(s) = \int_0^\infty e^{-st} f(t)dt$. Does the Laplace transform of $f(t) = \arctan t$ exist for $s > 0$? Explain your answer.

5. *(65 points)* Evaluate the following integrals:

$$\text{(a)}\ \int_{-1}^{1} \frac{3x+2}{\sqrt{4-x^2}}\,dx \qquad \text{(b)}\ \int_0^2 \frac{3x+2}{x^2-4x+3}\,dx$$

$$\text{(c)}\ \int \frac{x}{\sqrt{x^2+6x}}\,dx \qquad \text{(d)}\ \int_5^\infty \frac{\sqrt{x+4}}{x}\,dx$$

Some possibly useful formulas:

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 \\ \tan^2\theta + 1 &= \sec^2\theta \\ \sec^2\theta - 1 &= \tan^2\theta \end{aligned}$$

$$\begin{aligned} \sin 2A &= 2\sin A\cos A \\ \cos 2A &= \cos^2 A - \sin^2 A \\ &= 2\cos^2 A - 1 \\ &= 1 - 2\sin^2 A \end{aligned}$$

$$\int \tan x\,dx = \ln|\sec x| + C \qquad \int \sec x\,dx = \ln|\sec x + \tan x| + C$$

$$\int u\,dv = uv - \int v\,du \qquad (\overline{x}, \overline{y}) = \left(\frac{\int x(f(x)-g(x))dx}{\text{area}}\ ,\ \frac{\int \frac{1}{2}(f(x)^2 - g(x)^2)dx}{\text{area}} \right)$$

$$t = \tan\frac{x}{2}\ :\qquad \cos x = \frac{1-t^2}{1+t^2}\ ,\qquad \sin x = \frac{2t}{1+t^2}\ ,\qquad dx = \frac{2}{1+t^2}\,dt$$

Solutions to Exam II:

1. Because $\frac{d}{dx}(\arctan(2x)) = 2/(1+4x^2)$, the desired arc length is given by
$$\int_0^{\sqrt{3}/2} \sqrt{1+\frac{4}{(1+4x^2)^2}}\, dx \ .$$

2. The denominator is
$$(x-1)^2(x^2+3)^2 = (x^2-2x+1)(x^4+6x^2+9) = x^6-2x^5+7x^4-12x^3+15x^2-18x+9 \ ,$$
so long division
$$\begin{array}{r|l} & \quad 1 \\ \hline x^6-2x^5+7x^4-12x^3+15x^2-18x+9 & x^6+3x^5 \qquad\qquad - x^3 \qquad\qquad + 5x + 2 \\ & \underline{x^6-2x^5+7x^4-12x^3+15x^2-18x+9} \\ & \quad 5x^5-7x^4+11x^3-15x^2+23x-7 \end{array}$$
gives a quotient of 1 and a remainder of $5x^5-7x^4+11x^3-15x^2+23x-7$. Now x^2+3 is an irreducible (over the reals) quadratic, so the form of the partial fraction expansion is
$$\begin{aligned} \frac{x^6+3x^5-x^3+5x+2}{(x-1)^2(x^2+3)^2} &= 1+\frac{5x^5-7x^4+11x^3-15x^2+23x-7}{(x-1)^2(x^2+3)^2} \\ &= 1+\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{Cx+D}{x^2+3}+\frac{Ex+F}{(x^2+3)^2} \ . \end{aligned}$$

3. The area of the half-disc, which has radius a, is $\pi a^2/2$, and by symmetry the centroid must lie on the y-axis, so its x-coordinate is 0. So we only need to find its y-coordinate. Now the y-coordinate of any thin vertical slice is halfway up the slice, so
$$\begin{aligned} \overline{y} &= \frac{M_y}{\pi a^2/2} = \frac{2}{\pi a^2}\int_{-a}^{a} \frac{1}{2}\sqrt{a^2-x^2}\cdot\sqrt{a^2-x^2}\, dx \\ &= \frac{1}{\pi a^2}\int_{-a}^{a}(a^2-x^2)dx = \frac{1}{\pi a^2}\left[a^2x-\frac{1}{3}x^3\right]_{-a}^{a} = \frac{4a}{3\pi} \ . \end{aligned}$$

4. Because $\arctan t \geq 0$ for $t \geq 0$ and $\arctan t < \pi/2$ for all values of t, we have $0 \leq (\arctan t)e^{-st} \leq (\pi/2)e^{-st}$ for all t. Now for $s > 0$,
$$\int_0^\infty e^{-st}dt = \lim_{u\to\infty}\left(-\frac{1}{s}e^{-st}\Big|_0^u\right) = -\frac{1}{s}\lim_{u\to\infty}(e^{-su}-1) = \frac{1}{s} \ ,$$
so $(\pi/2)\int_0^\infty e^{-st}dt$ also converges (to $\pi/(2s)$), and hence, by the Comparison Test, the Laplace transform $L(s) = \int_0^\infty(\arctan x)e^{-st}dt$ exists (and is less than $< \pi/(2s)$), so we have a function that bounds the Laplace transform from above).

5. (a) The radical in the denominator suggests the substitution $x = 2\sin\theta$, so that $dx = 2\cos\theta\, d\theta$ and $\sqrt{4-x^2} = 2\cos\theta$, and as x varies from -1 to 1, θ varies from $-\arcsin(1/2)$ to $\arcsin(1/2)$ — let's denote that arcsin by α. So the integral becomes
$$\begin{aligned} \int_{-\alpha}^{\alpha}\frac{6\sin\theta+2}{2\cos\theta}2\cos\theta\, d\theta &= \int_{-\alpha}^{\alpha}(6\sin\theta+2)\, d\theta = [-6\cos\theta+2\theta]_{-\alpha}^{\alpha} \\ &= (-6\cos\alpha+2\alpha)-(-6\cos(-\alpha)+2(-\alpha)) \\ &= 4\alpha = 4\arcsin(1/2) \ , \end{aligned}$$

because the cosines of α and $-\alpha$ are equal.

(b) First, we should note that the denominator, $x^2 - 4x + 3 = (x-1)(x-3)$, is 0 at 1, inside the interval, so the integral is improper and may be divergent — we will need to find the limit of the antiderivative as x approaches 1 from both sides. We need to find the antiderivative: The form of the partial fraction expansion of the integrand is $(3x+2)/(x^2-4x+3) = A/(x-1) + B/(x-3)$. Multiplying both sides by x^2-4x+3 gives $3x+2 = A(x-3) + B(x-1) = (A+B)x + (-3A-B)$, or $A+B=3$ and $-3A-B=2$. Thus, $A = -5/2$ and $B = 11/2$:

$$\int \frac{3x+2}{x^2-4x+3}\,dx = \int \left(-\frac{5}{2}\,\frac{1}{x-1} + \frac{11}{2}\,\frac{1}{x-3}\right) dx = -\frac{5}{2}\ln|x-1| + \frac{11}{2}\ln|x-3| + C\ .$$

Now the limit as x approaches 1 (from either direction) of the first term is unbounded, so the improper integral is divergent.

(c) Because $x^2 + 6x = (x^2+6x+9) - 9 = (x+3)^2 - 9$, we set $x + 3 = 3\sec\theta$, so that $x = 3\sec\theta - 3$, $dx = 3\sec\theta\tan\theta\,d\theta$, and $\sqrt{x^2+6x} = 3\tan\theta$. So the integral becomes

$$\begin{aligned}
\int \frac{3\sec\theta - 3}{3\tan\theta}\,3\sec\theta\tan\theta\,d\theta &= \int \left(3\sec^2\theta - 3\sec\theta\right) d\theta \\
&= 3\tan\theta - 3\ln|\sec\theta + \tan\theta| + C \\
&= \sqrt{x^2+6x} - 3\ln\left|\frac{x+3}{3} + \frac{\sqrt{x^2+6x}}{3}\right| + C \\
&= \sqrt{x^2+6x} - 3\ln\left|x+3+\sqrt{x^2+6x}\right| + C_1\ ,
\end{aligned}$$

where $C_1 = C - \ln 3$.

(d) We first find the antiderivative of the function being integrated: Set $u = \sqrt{x+4}$, so that $x = u^2 - 4$ and $dx = 2u\,du$. Then the indefinite integral (without the limits) becomes

$$\int \frac{u}{u^2-4}\,2u\,du = \int \frac{2u^2}{u^2-4}\,du = 2\int \left(1 + \frac{4}{u^2-4}\right) du\ ,$$

the last step by long division. Now $4/(u^2-4) = A/(u-2) + B/(u+2)$ gives $4 = A(u+2) + B(u-2) = (A+B)u + (2A-2B)$, so $A = -B = 1$:

$$\begin{aligned}
2\int \left(1 + \frac{1}{u-2} - \frac{1}{u+2}\right) du &= 2(u + \ln|u-2| - \ln|u+2|) + C \\
&= 2\left(\sqrt{x+4} + \ln\left|\frac{\sqrt{x+4}-2}{\sqrt{x+4}+2}\right|\right) + C\ .
\end{aligned}$$

Now

$$\lim_{x\to\infty} \frac{\sqrt{x+4}-2}{\sqrt{x+4}+2} = \lim_{x\to\infty} \frac{1 - \frac{2}{\sqrt{x+4}}}{1 + \frac{2}{\sqrt{x+4}}} = \frac{1-0}{1+0} = 1\ ,$$

so the second term in the integral approaches 0 as $x \to \infty$; but the first term grows without bound, so the improper integral diverges.