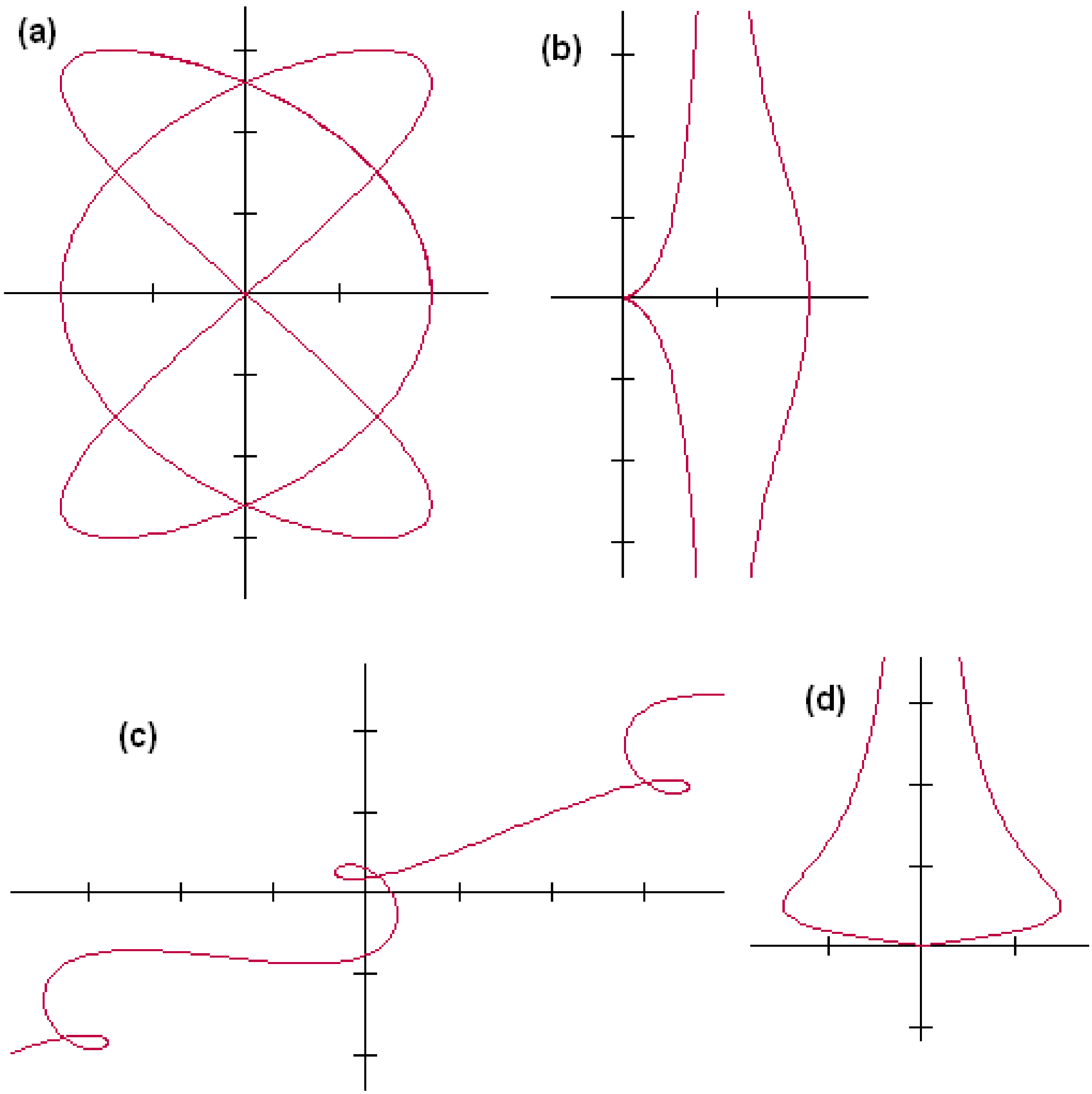
April 19, 2004

Math 112 — Practice Exam III

Show all work clearly; an answer with no justifying computations may not receive credit (except for the "write down but do not evaluate" questions).

1. *(13 points)* Which of the curves below are described by each of the given sets of parametric equations?

(i) $x = t + \sin(2t)$, $y = \dfrac{t}{2} + 0.3\cos(3t)$ (ii) $x = \dfrac{3t}{t^2+1}$, $y = \left|\dfrac{t}{2}\right|$



2. *(32 points)* Given the parametric equations

$$x = \cos^3\theta \ , \ \ y = \sin^3\theta \ \ :$$

(a) Find in terms of t the first and second derivatives dy/dx and d^2y/dx^2;

(b) At which points (x, y) is the tangent to the curve horizontal or vertical? (It's enough to consider θ-values in $[0, 2\pi]$.)

(c) On which interval or intervals (θ-values this time, in $[0, 2\pi]$) is the curve concave down?

(d) Find the area in the first quadrant under the curve.

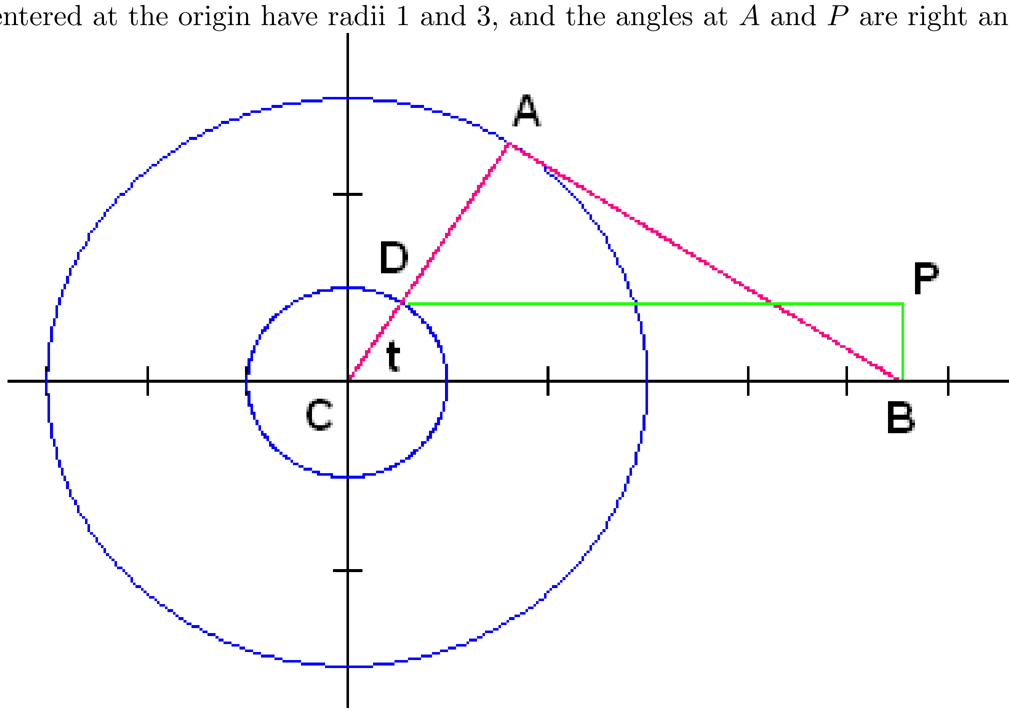
3. *(15 points)* Find the following Taylor series, centered at the points specified, as simply as possible:

$$\text{(a) } xe^{-x^2} \text{ at } x = 0 \qquad\qquad \text{(b) } \cos x \text{ at } x = \pi/2$$

4. *(30 points)* For which values of x does the following power series converge absolutely? (If it is a finite interval, tell what it does at the endpoints.)

$$\sum_{n=0}^{\infty} (-1)^n \frac{4^{n+2}}{3^{n+1}(2n+1)} (x-3)^n$$

5. *(10 points)* Find, in terms of the angle $\angle ACB = t$ the coordinates of the point P. The two circles centered at the origin have radii 1 and 3, and the angles at A and P are right angles.



Some possibly useful formulas:

$$\sum a_n \text{ converges if } \lim_{n\to\infty} \sqrt[n]{|a_n|} < 1 \text{ or } \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1.$$

$$a_n, b_n > 0, \lim_{n\to\infty} \frac{a_n}{b_n} \neq 0, \infty : \sum a_n \text{ converges } \iff \sum b_n \text{ converges}$$

$$\sum \frac{1}{n^p} \text{ converges } \Leftrightarrow p > 1 \qquad\qquad |r| < 1 : \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

$$a_n \text{ decr, } \lim a_n = 0 : \sum (-1)^n a_n \text{ converges; and } \left| \sum^{\infty} (-1)^n a_n - \sum^{N} (-1)^n a_n \right| < a_{N+1}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} \qquad\qquad s = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt \qquad\qquad \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

Solutions to Practice Exam III:

1. (i) Both the x- and y-values are unbounded, so this must be (c).
   (ii) The y-values are bounded below by 0, and the x-values are bounded between -3 and 3, so this must be (d).

2. (a) $dy/dx = (3\sin^2\theta\cos\theta)/(3\cos^2\theta(-\sin\theta)) = -\tan\theta$, and $d^2y/dx^2 = (-\sec^2\theta)/(-3\cos^2\theta\sin\theta) = 1/(3\cos^4\theta\sin\theta)$.
   (b) The tangent is horizontal when $dy/dx = 0$, i.e., when $\sin\theta = 0$, which occurs at $\theta = 0$ and π; the corresponding points are $(\pm 1, 0)$. It is vertical when dy/dx is undefined, i.e., when $\cos\theta = 0$, which occurs at $\theta = \pi/2$ and $3\pi/2$; the corresponding points are $(0, \pm 1)$.
   (c) The curve is concave down when $d^2y/dx^2 < 0$, which is true when $\sin\theta < 0$, which is true when $\pi < \theta < 2\pi$ (except at $\theta = -3\pi/2$, where the second derivative is undefined, because the cosine is 0).
   (d) As the x-values go from 0 to 1 (the largest x can be), the t-values go from $\pi/2$ to 0; so the desired area is

$$\begin{aligned}
\int_0^1 y\,dx &= \int_{\pi/2}^0 (\sin^3\theta)(3\cos^2\theta(-\sin\theta))d\theta = -3\int_{\pi/2}^0 \sin^4\theta\cos^2\theta\,d\theta \\
&= -\frac{3}{8}\int_{\pi/2}^0 (1-\cos 2\theta)^2(1+\cos 2\theta)d\theta \\
&= -\frac{3}{8}\int_{\pi/2}^0 (1-\cos 2\theta-\cos^2 2\theta+\cos^3 2\theta)d\theta \\
&= -\frac{3}{8}\int_{\pi/2}^0 (1-\cos 2\theta-\frac{1}{2}(1+\cos 4\theta)+(1-\sin^2 2\theta)\cos 2\theta)d\theta \\
&= -\frac{3}{8}\left[\theta-\frac{1}{2}\sin 2\theta-\frac{1}{2}(\theta+\frac{1}{4}\sin 4\theta)+\frac{1}{2}\sin 2\theta-\frac{1}{6}\sin^3 2\theta\right]_{\pi/2}^0 \\
&= 0+\frac{3}{8}\left(\frac{\pi}{2}-0-\frac{1}{2}\left(\frac{\pi}{2}+0\right)+0-0\right) = \frac{3\pi}{32}
\end{aligned}$$

3. (a) Because $e^x = \sum_{n=0}^\infty x^n/n!$, we have $xe^{-x^2} = x\sum_{n=0}^\infty(-x^2)^n/n! = \sum_{n=0}^\infty(-1)^n x^{2n+1}/n!$.
   (b) The derivatives cycle through $\cos x$, $-\sin x$, $-\cos x$, $\sin x$, so their values at $\pi/2$ cycle through $0, -1, 0, 1$. Dividing the n-th value by $n!$ and putting the results into the Taylor series formula gives

$$\begin{aligned}
\cos x &= 0-\frac{1}{1}(x-\frac{\pi}{2})-0+\frac{1}{3!}(x-\frac{\pi}{2})^3+0-\frac{1}{5!}(x-\frac{\pi}{2})^5-0+\ldots \\
&= -(x-\frac{\pi}{2})+\frac{1}{3!}(x-\frac{\pi}{2})^3-\frac{1}{5!}(x-\frac{\pi}{2})^5+\ldots = \sum_{n=1}^\infty \frac{(-1)^n}{(2n-1)!}(x-\frac{\pi}{2})^{2n-1}
\end{aligned}$$

4. Start with the Ratio Test:

$$\left|\frac{(-1)^{n+1}\dfrac{4^{n+3}}{3^{n+2}(2n+3)}(x-3)^{n+1}}{(-1)^n\dfrac{4^{n+2}}{3^{n+1}(2n+1)}(x-3)^n}\right| = \frac{4(2n+1)}{3(2n+3)}|x-3| \overset{n\to\infty}{\longrightarrow} \frac{4}{3}|x-3| < 1$$

when $|x-3| < 3/4$, i.e., $2\frac{1}{4} < x < 3\frac{3}{4}$, so on this interval the series converges, and for $x < 2\frac{1}{4}$ or $x > 3\frac{3}{4}$ it diverges. At $x = 2\frac{1}{4}$ and $x = 3\frac{3}{4}$, the series becomes

$$\sum_{n=0}^{\infty} \frac{16}{3(2n+1)} \qquad \text{and} \qquad \sum_{n=0}^{\infty} (-1)^n \frac{16}{3(2n+1)}$$

respectively. The first of these diverges, by Limit Comparison to the harmonic series (i.e., the p-series with $p = 1$), and the second converges by the Alternating Series Test; but the convergence is not absolute, because the sum of the absolute values of the terms is the first series, which, as we have just noted, diverges.

5. The y-coordinate of P is the same as the y-coordinate of D, i.e., $y = \sin t$. And the x-coordinate of P is equal to the x-coordinate of B, i.e., the hypotenuse of the right triangle with side $CA = 3$ adjacent to angle t; so $x = 3\sec t$.