Proofs of Convergence Tests for Series of Positive Terms

Suppose we have a series $\sum_{n=1}^{\infty} a_n$ when $a_n > 0$, and we want to know whether this series converges. Because the partial sums form an increasing sequence, the only way the series can fail to converge is by diverging to ∞; so we are really asking whether the partial sums are bounded above. Here are several tests for answering this question:

Integral Test: If $f(x)$ is a decreasing positive function from $[1, \infty)$ to $[0, \infty)$, then the series $\sum f(n)$ converges if and only if the improper integral $\int_0^\infty f(x)\,dx$ converges (i.e., is finite).

The (graphical) proof of this appears in a different supplement.

Comparison Test: If $0 < a_n \le b_n$ and $\sum b_n$ converges, then $\sum a_n$ also converges.

This is similar to the Comparison Test for improper integrals. The idea is simply that, because the partial sums of $\sum b_n$ are bounded above, then so are the smaller partial sums of $\sum a_n$.

Notes: (1) This statement is logically equivalent to its "contrapositive" statement: If $0 < a_n \le b_n$ and $\sum a_n$ diverges, then $\sum b_n$ also diverges.
(2) This test [respectively, the next test] is used as follows: Given a series $\sum a_n$, pick a series $\sum b_n$ about which it is known whether it converges or diverges, and check that $a_n < b_n$ [respectively, that $\lim_{n\to\infty}(a_n/b_n)$ is finite and nonzero].

Limit Comparison Test: If $0 < a_n, b_n$ and $\lim_{n\to\infty}(a_n/b_n) = L$, where L is neither 0 nor ∞, then either $\sum a_n$ and $\sum b_n$ both converge or both series diverge.

This is a mildly tricky application of the Comparison Test: Pick a number a bit larger than L — $L + 1$ will do. Because the quotient has limit L, for all n at least as large as some positive integer N we have $a_n/b_n < L + 1$, and so $a_n < (L + 1)b_n$. Now the series $\sum(L + 1)b_n$ converges (to $L + 1$ times whatever the series $\sum b_n$ converges to), so by the Comparison Test (applied only to the terms starting with the N-th one, but that doesn't affect convergence), we see that $\sum a_n$ also converges.

Note: The proof of this test shows that more is true. (It would just make the statement more complicated to try to include it.) For example, if $\lim(a_n/b_n) = 0$, then b_n is eventually larger than a_n; so if $\sum b_n$ converges, then so does $\sum a_n$. And if $\lim(a_n/b_n) = \infty$, then a_n is eventually larger than b_n, so if $\sum b_n$ diverges, then so does a_n. But for neither of these statements is the converse true.

Ratio Test: If $0 < a_n$ and $\lim_{n\to\infty}(a_{n+1}/a_n) < 1$, then $\sum a_n$ converges. If the limit is greater than 1, then the series diverges. And if the limit is equal to 1, then the test gives no information.

This application of the Comparison Test is trickier still: Suppose the limit is less than 1, and pick a number r larger than the limit but still less than 1 (for instance, the average of the limit and 1). Then because the limit is less than r for all n at least as large as some positive integer N we have $a_{n+1}/a_n < r$, i.e., $a_{n+1} < a_n r$. So for all $k \ge 1$ we have

$$a_{N+k} < a_{N+k-1}r < (a_{N+k-2}r)r = a_{N+k-2}r^2 < (a_{N+k-3}r)r^2 = a_{N+k-3}r^3 < \ldots < a_N r^k \ .$$

So starting with the N-th term, the terms of the series $\sum a_n$ are bounded above by the terms of the geometric series $\sum a_N r^k$ (where the last sum runs over k, not n). This geometric series has common ratio $r < 1$, so it converges; and hence, so does $\sum a_n$.
Suppose the initial limit were greater than 1. Then in a similar way we can see that the terms a_n are eventually bounded below by the terms of a a geometric series with common ratio greater than 1, so $\sum a_n$ diverges.

Note: This test is easy to use, but it often gives a limit of 1, so it is often inconclusive. For example, the Integral Test shows that the p-series $\sum 1/n^p$ converges for $p > 1$ and diverges for $p \le 1$, but the Ratio

Test gives a limit of 1 in both cases.

Root Test: If $0 < a_n$ and $\lim_{n\to\infty} \sqrt[n]{a_n} < 1$, then $\sum a_n$ converges. If the limit is greater than 1, then the series diverges. And if the limit is equal to 1, then the test gives no information.

Suppose the limit is less than 1, and pick a number r between the limit and 1. Then for all "sufficiently large" n we have $\sqrt[n]{a_n} < r$, i.e., $a_n < r^n$. Thus, for all sufficiently large n the terms in the given series are bounded above by the terms of the convergent geometric series $\sum r^n$, so the given series also converges.

On the other hand, if the limit is greater than 1 and we choose an r between the limit and 1, then the terms of the given series are eventually greater than the terms in the divergent geometric series $\sum r^n$, so the given series diverges.

Note: The Root Test is harder to apply than the Ratio Test, but it can sometimes show convergence or divergence when the Ratio Test is inconclusive. Like the Ratio Test, however, it is inconclusive on p-series (and hence also on the many series that could be decided with the Limit Comparison Test applied with p-series).