Here is a proof of (part of) L'Hôpital's Rule. It is stated for the case where x approaches a from above for two reasons: (1) proving it with a weaker hypothesis makes a stronger theorem (and the other cases — x approaching a from below or from both sides — are easy to get from this case, anyway), and (2) it allows us to avoid some of the absolute value bars in the definition of limit: instead of $|x-a|<\delta$, we have $0<x-a<\delta$. We begin with a stronger version of the Mean Value Theorem than is proved in the main section of Stewart. He does state this stronger version in Appendix F, and the proof is sketched. (Moreover, what is given here isn't much longer.)

Cauchy's (Extended) Mean Value Theorem: Suppose f and g are functions continuous on a closed interval $[a,b]$ and differentiable on (a,b). Then there is a value c in (a,b) for which

$$f'(c)(g(b)-g(a)) = g'(c)(f(b)-f(a)) \ .$$

Proof: Define the function h on $[a,b]$ by

$$h(x) = f(x)(g(b)-g(a)) - g(x)(f(b)-f(a)) \ .$$

Then h is continuous on $[a,b]$ and differentiable on (a,b), and $h(a) = f(a)g(b) - g(a)f(b) = h(b)$, so by Rolle's Theorem, there is a c in (a,b) for which $h'(c)=0$. Since $h'(x) = f'(x)(g(b)-g(a)) - g'(x)(f(b)-f(a))$, the result follows. //

L'Hôpital's Rule: Suppose either

$$\lim_{x\to a^+} f(x) = 0 = \lim_{x\to a^+} g(x) \qquad (1)$$

or

$$\lim_{x\to a^+} g(x) = \infty \ . \qquad (2)$$

Then

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)} \ .$$

Proof: The proof in the case of hypotheses (1) appears in Appendix F of Stewart's text, so it won't be repeated here; so we assume hypothesis (2). Write L for $\lim_{x\to a^+}(f'(x)/g'(x))$, and let $\varepsilon>0$ be given. Pick $\delta_1>0$ so that $0<x-a<\delta_1$ implies, first, that $g(x)>0$ (which is possible, since $g(x)\to\infty$) and, second, that

$$\left|\frac{f'(x)}{g'(x)} - L\right| < \frac{\varepsilon}{2} \ .$$

Then for $a<x<x_1<a+\delta_1$, by Cauchy's Mean Value Theorem there is a value c in (x,x_1) for which

$$\frac{f(x)-f(x_1)}{g(x)-g(x_1)} = \frac{f'(c)}{g'(c)} \ , \qquad \text{and hence} \qquad \left|\frac{f(x)-f(x_1)}{g(x)-g(x_1)} - L\right| < \frac{\varepsilon}{2}$$

(and that is the last use we make of c, f' and g'). Rewriting the last inequality as two inequalities, we get

$$L - \frac{\varepsilon}{2} < \frac{f(x)-f(x_1)}{g(x)-g(x_1)} < L + \frac{\varepsilon}{2} \ ,$$

and multiplying through by $(g(x)-g(x_1))/g(x) = 1-(g(x_1)/g(x))$ and adding $f(x_1)/g(x)$ throughout gives

$$\left(L-\frac{\varepsilon}{2}\right)\left(1-\frac{g(x_1)}{g(x)}\right) + \frac{f(x_1)}{g(x)} < \frac{f(x)}{g(x)} < \left(L+\frac{\varepsilon}{2}\right)\left(1-\frac{g(x_1)}{g(x)}\right) + \frac{f(x_1)}{g(x)} \ .$$

Pick x_1 in $(a, a+\delta_1)$ and keep it fixed. Then as $x \to a^+$ and $g(x) \to \infty$, we have that $g(x_1)/g(x)$ and $f(x_1)/g(x)$ both approach 0, and so we know the limits of the left and right ends of the last inequality:

$$\left(L - \frac{\varepsilon}{2}\right)\left(1 - \frac{g(x_1)}{g(x)}\right) + \frac{f(x_1)}{g(x)} \to L - \frac{\varepsilon}{2}$$

and

$$\left(L + \frac{\varepsilon}{2}\right)\left(1 - \frac{g(x_1)}{g(x)}\right) + \frac{f(x_1)}{g(x)} \to L + \frac{\varepsilon}{2} \ .$$

So there is a $\delta > 0$ such that $\delta < x_1 - a$ and if $0 < x - a < \delta$, then

$$\left(L - \frac{\varepsilon}{2}\right)\left(1 - \frac{g(x_1)}{g(x)}\right) + \frac{f(x_1)}{g(x)} > L - \varepsilon \qquad \text{and} \qquad \left(L + \frac{\varepsilon}{2}\right)\left(1 - \frac{g(x_1)}{g(x)}\right) + \frac{f(x_1)}{g(x)} < L + \varepsilon \ .$$

Then $0 < x - a < \delta$ implies

$$L - \varepsilon < \frac{f(x)}{g(x)} < L + \varepsilon \ , \qquad \text{i.e.,} \qquad \left|\frac{f(x)}{g(x)} - L\right| < \varepsilon \ .$$

Therefore, $\lim_{x \to a^+}(f(x)/g(x)) = L$. //