1. (from Stewart, page 586) Solve the initial value problem.
$$\frac{du}{dt} = \frac{2t + \sec^2 t}{2u} , \qquad u(0) = -5 .$$

2. (from Stewart, page 586)
   (a) Solve $y' = 2x\sqrt{1-y^2}$.
   (b) Solve $y' = 2x\sqrt{1-y^2}$, $y(0) = 0$.
   (c) Solve $y' = 2x\sqrt{1-y^2}$, $y(0) = 2$ if possible.

3. $\dfrac{dy}{dx} = \dfrac{y+1}{2xy}$

4. $x \cos^2 y + \tan y \dfrac{dy}{dx} = 0$

5. $2xyy' = 1 + y^2$, $y(2) = 3$

6. $1 + \ln x = -(1 + \ln y)y'$

7. Use $u = xy$ to solve $y' = xy + y + 1 + \dfrac{1-y}{x}$.

8. A weight at $(0, a)$ on the coordinate plane is attached to a taut rope, the other end of which is held by by a man at the origin. The man walks down the positive x-axis, pulling the weight. Find the path of the weight, which is called a *tractrix.*

9. A thermometer reading 75°F is taken outdoors where the temperature is 20°F. The reading is 30°F four minutes later. Find
   (a) the thermometer reading after seven minutes outdoors, and
   (b) the time outdoors until the reading is 21°F.

10. A tank contains 100 gal of salt water at a concentration of 6 lb/gal. Salt water at a concentration of 1 lb/gal is pumped into the tank at a rate of 2 gal/min, and the well-stirred mixture runs out at the rate of 2 gal/min. How much salt is in the tank after t min?

11. (from Stewart, page 587) A certain small country has \$10 B in paper currency in circulation, and each day \$50 M comes into the country's banks. The government decides to introduce new currency by having the banks replace the old bills with new ones whenever old currency comes

into the banks. Let $x = x(t)$ denote the amount of new currency in circulation at time t (in units of \$10 M), with $x(0) = 0$.

(a) Formulate a mathematical model in the form of an initial value problem that represents the "flow" of the new currency into circulation.

(b) Solve the initial value problem in part (a).

(c) How long with it take for the new bills to account for 90% of the currency in circulation?

Solutions

1.

$$\begin{aligned}
\int 2u\,du &= \int (2t + \sec^2 t)\,dt \\
u^2 &= t^2 + \tan t \ + C \\
(-5)^2 &= 0^2 + \tan 0 + C \qquad \Longrightarrow \qquad C = 25 \\
u &= \pm\sqrt{t^2 + \tan t + 25}
\end{aligned}$$

2. (a)

$$\begin{aligned}
\int \frac{dy}{\sqrt{1-y^2}} &= \int 2x\,dx \\
\arcsin y &= x^2 + C \\
y &= \sin(x^2 + C)
\end{aligned}$$

(b) $0 = \sin(0^2 + C)$ gives $C = 0$ (or any integer multiple of π).

(c) $2 = \sin(0^2 + C)$ has not solution, so there is no solution to this initial value problem.

3.

$$\begin{aligned}
\int \frac{2y\,dy}{y+1} &= \int \frac{dx}{x} \\
\int \left(2 - \frac{2}{y+1}\right) dy &= \int \frac{dx}{x} \\
2y - 2\ln|y+1| &= \ln|x| + C \\
\frac{e^{2y}}{(y+1)^2} &= |x| \cdot e^C \\
e^{2y} &= Kx(y+1)^2
\end{aligned}$$

4.

$$\begin{aligned}
x\cos^2 y &= -\tan y \frac{dy}{dx} \\
\int x\,dx &= -\int \frac{\tan y}{\cos^2 y}\,dy = \int \frac{-\sin y\,dy}{\cos^3 y} \\
\frac{1}{2}x^2 &= -\frac{1}{2}\cos^{-2} y + C = -\sec^2 y + C
\end{aligned}$$

5.

$$\begin{aligned}
\int \frac{2y\,dy}{1+y^2} &= \int \frac{dx}{x} \\
\ln|1+y^2| &= \ln|x| + C \\
1 + y^2 &= Kx \\
1 + 3^2 &= K2 \qquad \Longrightarrow \qquad K = 2 \\
1 + y^2 &= 2x \\
y &= \pm\sqrt{2x-1}
\end{aligned}$$

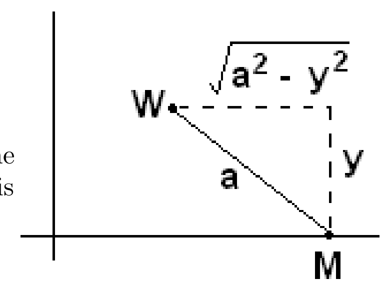
6.

$$\begin{aligned}
\int (1+\ln x)\,dx &= -\int (1+\ln y)\,dy \\
x + x\ln x - x &= -y - y\ln y + y + C \qquad \text{[Table \#100]} \\
x\ln x &= -y\ln y + C
\end{aligned}$$

7. If the equation is separable as it stands, it is not obvious; so we use the substitution suggested in the problem: Because $u = xy$, $y = u/x$, so $y' = (xu' - u)/x^2$, the equation becomes

$$\begin{aligned}
\frac{xu'-u}{x^2} &= x\left(\frac{u}{x}\right) + \frac{u}{x} + 1 + \frac{1-u/x}{x} \\
\frac{u'}{x} - \frac{u}{x^2} &= u + \frac{u}{x} + 1 + \frac{x-u}{x^2} \\
u' &= x\left(u + \frac{u}{x} + 1 + \frac{1}{x}\right) = ux + u + x + 1 = (u+1)(x+1) \\
\int \frac{du}{u+1} &= \int (x+1)\,dx \\
\ln|u+1| &= \frac{1}{2}x^2 + x + C \\
u &= -1 + Ke^{x^2/2+x}
\end{aligned}$$

8. If the weight is at point (x, y), the man is at a distance of a and the weight is being pulled in his direction, so its direction of motion is toward the man, and hence the slope of its path is $-y/\sqrt{a^2-y^2}$.



The resulting integral calls for trig substitution: $y = a\sin\theta$, $dy = a\cos\theta\,d\theta$, and $\sqrt{a^2-y^2} = a\cos\theta$:

$$\begin{aligned}
y' &= \frac{-y}{\sqrt{a^2-y^2}} \\
\int \frac{\sqrt{a^2-y^2}\,dy}{y} &= -\int dx \\
-x + C &= \int \frac{(a\cos\theta)(a\cos\theta\,d\theta)}{a\sin\theta} = a\int \frac{(1-\sin^2\theta)\,d\theta}{\sin\theta} \\
&= a\int (\csc\theta - \sin\theta)\,d\theta = a\ln|\csc\theta - \cot\theta| + a\cos\theta \\
&= a\ln\left|\frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta}\right| + a\cos\theta = a\ln\left|\frac{a}{y} - \frac{\sqrt{a^2-y^2}}{y}\right| + \sqrt{a^2-y^2} \\
&= a\ln(a - \sqrt{a^2-y^2}) - a\ln y + \sqrt{a^2-y^2}\ ,
\end{aligned}$$

where we have used the facts that $y > 0$ and $\sqrt{a^2-y^2} \le a$. so

$$x = a\ln y - a\ln(a - \sqrt{a^2-y^2}) - \sqrt{a^2-y^2} + K$$

Now when $x = 0$ we have $y = a$ and hence $\sqrt{a^2-y^2} = 0$, so

$$0 = a\ln a - a\ln(a-0) - \sqrt{0} + K \qquad \Longrightarrow \qquad K = 0\ .$$

So the equation of the tractrix is $x = a\ln y - a\ln(a - \sqrt{a^2 - y^2}) - \sqrt{a^2 - y^2}$. Note that

$$\begin{aligned}
\ln y - \ln(a - \sqrt{a^2 - y^2}\,) &= \ln\left|\frac{y}{a - \sqrt{a^2 - y^2}}\right| = \ln\left|\frac{y(a + \sqrt{a^2 - y^2}\,)}{a^2 - (a^2 - y^2)}\right| \\
&= \ln\left|\frac{a + \sqrt{a^2 - y^2}}{y}\right| ,
\end{aligned}$$

so the equation of the tractrix can also be expressed in the form

$$x = a\ln\left|\frac{a + \sqrt{a^2 - y^2}}{y}\right| - \sqrt{a^2 - y^2}\ .$$

9. Let T be the thermometer's temperature reading in degrees Farenheit after t minutes outdoors. Then $T(0) = 75$ and $T(4) = 30$, and using Newton's Law of Cooling, we have $dT/dt = k(T - 20)$ for some negative constant k. From $\int dT/(T - 20) = \int k\,dt$ we get $\ln(T - 20) = kt + C$ for some constant C. (We don't need to write $\ln|T - 20|$ because T will always be at least 20.) Now the initial conditions give

$$\begin{aligned}
\ln(75 - 20) &= k \cdot 0 + C &\Longrightarrow&\quad C = \ln(55) \\
\ln(30 - 20) &= k \cdot 4 + \ln(55) &\Longrightarrow&\quad k = \frac{1}{4}\ln\left(\frac{10}{55}\right)
\end{aligned}$$

(Note that $10/55 < 1$, so that k really is negative.) So we have

$$T = 20 + 55e^{(t/4)\ln(10/55)}\ .$$

   (a) When $t = 7$, $T = 20 + 55e^{(-7/4)\ln(55/10)} \approx 22.8$.

   (b) For $T = 21$, we get

$$1 = 55e^{(t/4)\ln(10/55)} \qquad \Longrightarrow \qquad t = 4\frac{\ln(1/55)}{\ln(10/55)} \approx 9.4\ .$$

10. Let s denotes the number of lb of salt in the tank after t minutes. Then the 2 gal of salt water running out of the tank each minute take with it $2s/100$ lb of salt. So a differential equation for s is

$$\frac{ds}{dt} = 2 - \frac{2s}{100}\ .$$

Note that $s(0) = 600$ as before, and also that, because the water entering the tank has 1 lb/gal of salt, s is never less than 100. Now this differential equation is separable: $ds/dt = (200 - 2s)/100 = -(s - 100)/50$, so $\int dx/(s - 100) = \int(-1/50)dt$ gives $\ln(s - 100) = -t/50 + C$, or $s = 100 + e^C \cdot e^{-t/50}$. And the initial condition gives $e^C = 500$, so $s = 100 + 500e^{-t/50}$.

11. (a) Because 1000 units of paper currency in circulation at any time, when there are x new currency in circulation, it constitutes $x/1000$ of all currency; so we may assume that $(1 - \frac{x}{1000})5$ of the currency that enters the banks in a day is old currency, which will be replaced by new currency. Thus, the rate of change in x is $(1 - \frac{x}{1000})5$, i.e., the desired equation is

$$\frac{dx}{dt} = 5 - \frac{x}{200}\ .$$

   And of course the initial value of x, when $t = 0$, is 0.

   (b) $\int 200\,dx/(1000 - x) = dt$ gives $-200\ln(1000 - x) = t + C$, and from the initial value we get $C = -200\ln 1000$. So $t = 200\ln\frac{1000}{1000-x}$. (Probably it would be more reasonable to solve for x in terms of t rather than the other way around, but this version gives the answer to (c) more easily.)

   (c) We get $x = 900$ when $t = 200\ln 10$, which my calculator says is about 461 days.