Deriving the Second Derivative Test

Start with $z = f(x, y)$ and a point (a, b) for which $f_x(a, b) = f_y(a, b) = 0$. To simplify notation, let's write

$$f_{xx}(a,b) = T, \qquad f_{xy}(a,b) = U, \qquad f_{yy}(a,b) = V.$$

Then using the second-degree Taylor polynomial for f gives

$$\begin{aligned} f &\approx f(a,b) + \frac{T}{2}(x-a)^2 + U(x-a)(y-b) + \frac{V}{2}(y-b)^2 \\ &= f(a,b) + \frac{T}{2}\left[(x-a)^2 + 2\frac{U}{T}(x-a)(y-b) + \left(\frac{U}{T}\right)^2 (y-b)^2 \right. \\ &\qquad \left. + \left(\frac{V}{T} - \frac{U^2}{T^2}\right)(y-b)^2\right] \\ &= f(a,b) + \frac{T}{2}\left[\left\{(x-a) + \frac{U}{T}(y-b)\right\}^2 + \frac{TV-U^2}{T^2}(y-b)^2\right] \end{aligned}$$

Thus, if $TV - U^2 < 0$, the two square terms have opposite signs, so f resembles a hyperbolic parabolic — (a, b) is a saddle point.

And if $TV - U^2 > 0$, it resembles an elliptical paraboloid, opening upward if $T > 0$ and downward if $T < 0$.

But if $TV - U^2 = 0$, the second-degree Taylor polynomial doesn't give enough information to classify the critical point (a, b) — compare $x^4 + y^4$ and $x^3 - xy^2$, one with a local minimum at $(0, 0)$ and the other with a saddle point there, but both with $TV - U^2 = 0$.

Therefore: Suppose $f_x(a, b) = f_y(a, b) = 0$, and set $D = f_{xx}(a, b) f_{yy}(a, b) - f_{xy}(a, b)^2$. Then:

- if $D < 0$, then f has a saddle point at (a, b);
- if $D > 0$ and $f_{xx}(a, b) > 0$, then f has a local minimum at (a, b);
- if $D > 0$ and $f_{xx}(a, b) < 0$, then f has a local maximum at (a, b); and
- if $D = 0$, anything can happen.