Deriving the Taylor polynomial (in one variable)

Start with $y = f(x)$ and a point $x = a$. We want a polynomial

$$P(x) = C_0 + C_1(x-a) + C_2(x-a)^2 + C_3(x-a)^3$$

for which

$$P(a) = f(a), \qquad P'(a) = f'(a), \qquad P''(a) = f''(a), \qquad P'''(a) = f'''(a);$$

i.e., P is the best approximation to f near a by a polynomial of degree at most 3. What must the coefficients C_0, C_1, C_2 and C_3 be?

$$\begin{array}{ll}
P(x) = C_0 + C_1(x-a) + C_2(x-a)^2 + C_3(x-a)^3 & P(a) = C_0 \\
P'(x) = C_1 + 2C_2(x-a) + 3C_3(x-a)^2 & P'(a) = C_1 \\
P''(x) = 2C_2 + 6C_3(x-a) & P''(a) = 2C_2 \\
P'''(x) = 6C_3 & P'''(a) = 6C_3
\end{array}$$

So we must have $C_0 = f(a)$, $C_1 = f'(a)$, $C_2 = \frac{1}{2}f''(a)$, and $C_3 = \frac{1}{6}f'''(a)$ to get what we want; i.e.,

$$P(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \frac{1}{6}f'''(a)(x-a)^3$$

In general, if we want a polynomial of degree N that has the same first $N+1$ derivatives (including the 0-th derivative, i.e., f itself) as f at a, it would be

$$\sum_{n=0}^{N} \frac{1}{n!} f^{(n)}(a)(x-a)^n \ .$$