

A change of variables from x, y to s, t turns the rectangle with sides $ds \vec{i}$, $dt \vec{j}$ into the parallelogram with sides

$$\frac{\partial x}{\partial s} ds \vec{i} + \frac{\partial y}{\partial s} ds \vec{j}, \quad \frac{\partial x}{\partial t} dt \vec{i} + \frac{\partial y}{\partial t} dt \vec{j};$$

(or at least the image of the rectangle differs in a negligible way from the parallelogram, for small enough ds and dt) and the area of this parallelogram is the length of the cross product of these two sides, i.e.,

$$\begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{vmatrix} ds dt .$$

(The area is regarded as negative if the smaller angle from the image of \vec{i} to that of \vec{j} is clockwise; the length of the cross product isn't negative, of course, but the determinant of partial derivatives may be, and its sign agrees with this idea of positive and negative areas.) The determinant of partial derivatives is called the *Jacobian* of the change of variables and can be denoted

$$\mathcal{J}(x, y; s, t) .$$

Note that we also have

$$\mathcal{J}(x, y; s, t) = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix} .$$

Because $\mathcal{J}(x, y; s, t)$ is the factor that changes an (infinitesimal) increment of area in the st -plane into the corresponding increment of area in the xy -plane, we have the following change of variables formula for double integrals: Let R be a region in the xy -plane, and let W be a region in the st -plane that maps onto R by the change of variables. Then

$$\int_R f(x, y) dx dy = \int_W f(x(s, t), y(s, t)) \mathcal{J}(x, y; s, t) ds dt .$$

For example, because the polar coordinate formulas are $x = r \cos \theta$ and $y = r \sin \theta$, we have

$$\mathcal{J}(x, y; r, \theta) = \begin{vmatrix} \cos \theta & \sin \theta \\ -r \sin \theta & r \cos \theta \end{vmatrix} = r \cos^2 \theta - (-r \sin^2 \theta) = r ,$$

so the increment of area in polar coordinates is $r dr d\theta$.

Similarly, in three dimensions, suppose we have a change of variables from x, y, z to s, t, u . Then the rectangular parallelepiped determined by $ds \vec{i}$, $dt \vec{j}$, $du \vec{k}$ is taken to the (probably not rectangular) parallelepiped determined by

$$\begin{aligned} \frac{\partial x}{\partial s} ds \vec{i} + \frac{\partial y}{\partial s} ds \vec{j} + \frac{\partial z}{\partial s} ds \vec{k} , & \quad \frac{\partial x}{\partial t} dt \vec{i} + \frac{\partial y}{\partial t} dt \vec{j} + \frac{\partial z}{\partial t} dt \vec{k} , \\ & \quad \frac{\partial x}{\partial u} du \vec{i} + \frac{\partial y}{\partial u} du \vec{j} + \frac{\partial z}{\partial u} du \vec{k} ; \end{aligned}$$

and this parallelepiped has volume the triple product of these vectors, i.e.,

$$\begin{vmatrix} \frac{\partial x}{\partial s} ds & \frac{\partial x}{\partial t} dt & \frac{\partial x}{\partial u} du \\ \frac{\partial y}{\partial s} ds & \frac{\partial y}{\partial t} dt & \frac{\partial y}{\partial u} du \\ \frac{\partial z}{\partial s} ds & \frac{\partial z}{\partial t} dt & \frac{\partial z}{\partial u} du \end{vmatrix} = \mathcal{J}(x, y, z; s, t, u) ds dt du .$$

So the determinant of partial derivatives, again called the Jacobian, gives the factor that changes an increment of volume in stu -space into an increment of volume in xyz -space. And we have the following change of variables formula for triple integrals: Let R be a region in xyz -space, and let W be a region in stu -space that maps onto R by the change of variables. Then

$$\begin{aligned} \int_R f(x, y, z) dx dy dz \\ = \int_W f(x(s, t, u), y(s, t, u), z(s, t, u)) \mathcal{J}(x, y, z; s, t, u) ds dt du . \end{aligned}$$

For example, the cylindrical coordinate formulas are $x = r \cos \theta$, $y = r \sin \theta$ and $z = z$, and the spherical coordinates formulas are $x = \rho \sin \phi \cos \theta$, $y = \rho \sin \phi \sin \theta$ and $z = \rho \cos \phi$. So we have

$$\mathcal{J}(x, y, z; r, \theta, z) = \begin{vmatrix} \cos \theta & \sin \theta & 0 \\ -r \sin \theta & r \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (r \cos^2 \theta - (-r \sin^2 \theta))1 = r ,$$

$$\mathcal{J}(x, y, z; \rho, \phi, \theta) = \begin{vmatrix} \sin \phi \cos \theta & \sin \phi \sin \theta & \cos \phi \\ \rho \cos \phi \cos \theta & \rho \cos \phi \sin \theta & -\rho \sin \phi \\ -\rho \sin \phi \sin \theta & \rho \sin \phi \cos \theta & 0 \end{vmatrix} = \rho^2 \sin \phi ,$$

so the increments of volume in cylindrical and spherical coordinates are $r dr d\theta dz$ and $\rho^2 \sin \phi dr d\phi d\theta$ respectively.