Derivation of the Directional Derivative Formula

We let $\vec{u} = u_1\vec{\imath} + u_2\vec{\jmath}$ be a fixed unit vector in the xy-plane, and as usual, we let $z = f(x, y)$. We want to show that the simple formula for the directional derivative

$$f_{\vec{u}} = f_x u_1 + f_y u_2$$

is valid wherever f is a differentiable function. We fix a particular point (a, b) in the xy-plane where f is differentiable, and we recall that the formal definition of $f_u(a, b)$ is

$$f_u(a, b) = \lim_{h\to 0} \frac{f(a + u_1h, b + u_2h) - f(a, b)}{h} \ ;$$

so we want to show that the limit simplifies to

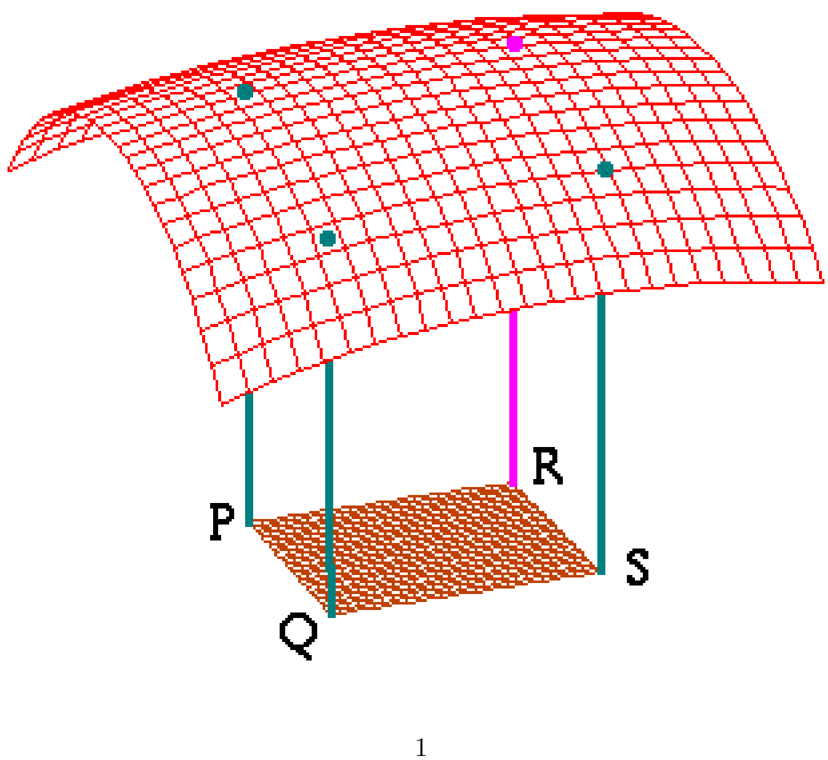
$$f_x(a, b)u_1 + f_y(a, b)u_2 \ . \quad (*)$$

Now we can certainly rewrite the difference quotient in the definition in the following form, dividing it into one part where the x-value varies while y remains fixed and the other where the reverse holds:

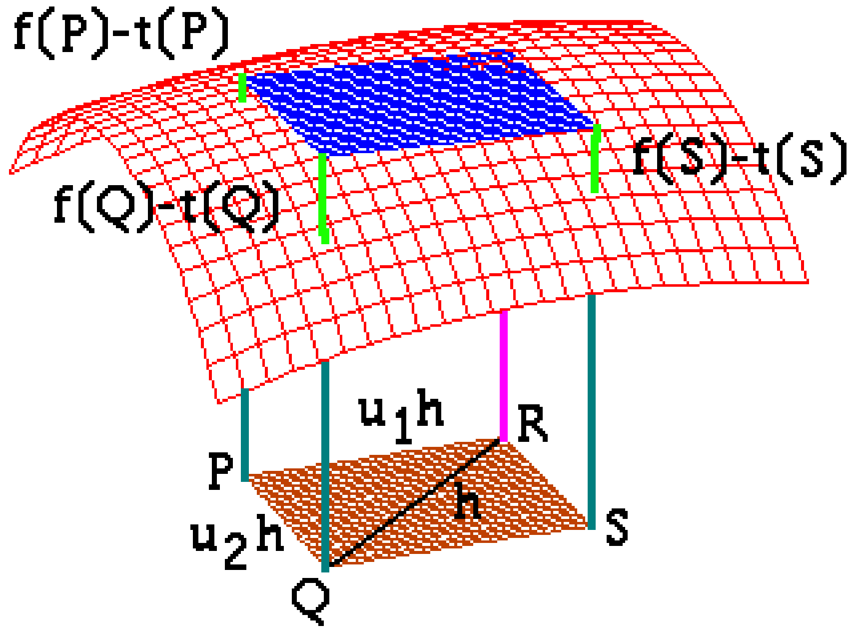
$$\frac{f(a + u_1h, b + u_2h) - f(a, b)}{h} = \frac{f(a + u_1h, b + u_2h) - f(a, b + u_2h)}{u_1h}u_1 + \frac{f(a, b + u_2h) - f(a, b)}{u_2h}u_2 \ . \quad (**)$$

Because the change in the x-value in the first term is u_1h, we have arranged for the denominator to have that value, in hopes that it will approach the partial derivative of f with respect to x; and similarly for y in the second term. Now as $h \to 0$, we also have $u_2h \to 0$, so the fraction $[f(a, b + u_2h) - f(a, b)]/(u_2h)$ in the second term goes to $f_y(a, b)$. And the factors u_1 and u_2 are constants. That leaves the fraction $[f(a + u_1h, b + u_2h) - f(a, b + u_2h)]/(u_1h)$, which we need to show goes to $f_x(a, b)$ as $h \to 0$. (Later we will want to divide by u_1, so let us remark now that if $u_1 = 0$, we already know that the formula $(*)$ holds, because the first term is 0. So for the rest of the proof we may safely assume that $u_1 \neq 0$.)

Let us give labels to some points in the xy-plane: $P(a + u_1h, b)$, $Q(a + u_1h, b + u_2h)$, $R(a, b)$ and $S(a, b + u_2h)$. Also, if $T(c, d)$ is any point in the xy-plane, we will write $f(T)$ in place of $f(c, d)$. Then as $h \to 0$, P, Q and S all slide towards R, and the limit of the quotient $(f(S) - f(R))/(u_1h)$ is $f_x(a, b)$ by definition. But the numerator of the difference quotient we are studying comes from the other side of the rectangle: $[f(a + u_1h, b + u_2h) - f(a, b + u_2h)]/(u_1h) = (f(Q) - f(P))/(u_1h)$.



To show that these two have the same limit, we need to use the full strength of differentiability, as discussed in Section 13.10 of the text: We let $t(x,y)$ denote the tangent plane to the graph of f at $R(a,b)$.



Differentiability says that, as a point T (in the xy-plane) approaches R, the graph of f is so close to its tangent plane that the quotient $(f(T)-t(T))/(TR)$ approaches 0 — though its denominator, the distance from T to R, is approaching 0, the numerator, the vertical distance between surface and tangent plane at T, is approaching 0 even faster.

We want to show that the difference between $(f(Q)-f(P))/(u_1h)$ (which we *want* to know approaches $f_x(a,b)$ as $h \to 0$) and $(f(S)-f(R))/(u_1h)$ (which we *know* approaches $f_x(a,b)$) approaches 0. So we rewrite that difference by breaking its numerator into parts: the values of the corresponding differences $t(Q)-t(P)$ and $t(S)-t(R)$ on the tangent plane (which are equal, because it *is* a plane and QP and SR are on opposite sides of a rectangle) and the vertical distances between the tangent plane and the surface f. The vertical distance at R is 0, because the plane is tangent there. So we get:

$$\begin{aligned}\frac{f(Q)-f(P)}{u_1h}-\frac{f(S)-f(R)}{u_1h} &= \frac{t(Q)-t(P)}{u_1h}-\frac{t(S)-t(R)}{u_1h}+\frac{f(Q)-t(Q)}{u_1h}-\frac{f(P)-t(P)}{u_1h}-\frac{f(S)-t(S)}{u_1h}\\ &= 0+\frac{f(Q)-t(Q)}{h}\left(\frac{1}{u_1}\right)-\frac{f(P)-t(P)}{u_1h}-\frac{f(S)-t(S)}{u_2h}\left(\frac{u_2}{u_1}\right)\end{aligned}$$

Because the distances from Q, P and S to R are h, u_1h and u_2h respectively, the last three terms approach 0 as $h \to 0$ by differentiability, so

$$\lim_{h\to 0}\frac{f(Q)-f(P)}{u_1h}=\lim_{h\to 0}\frac{f(S)-f(R)}{u_1h}=f_x(a,b)\ .$$

Thus, taking the limit as $h \to 0$ in $(**)$ gives the desired formula $(*)$ and the proof is complete.