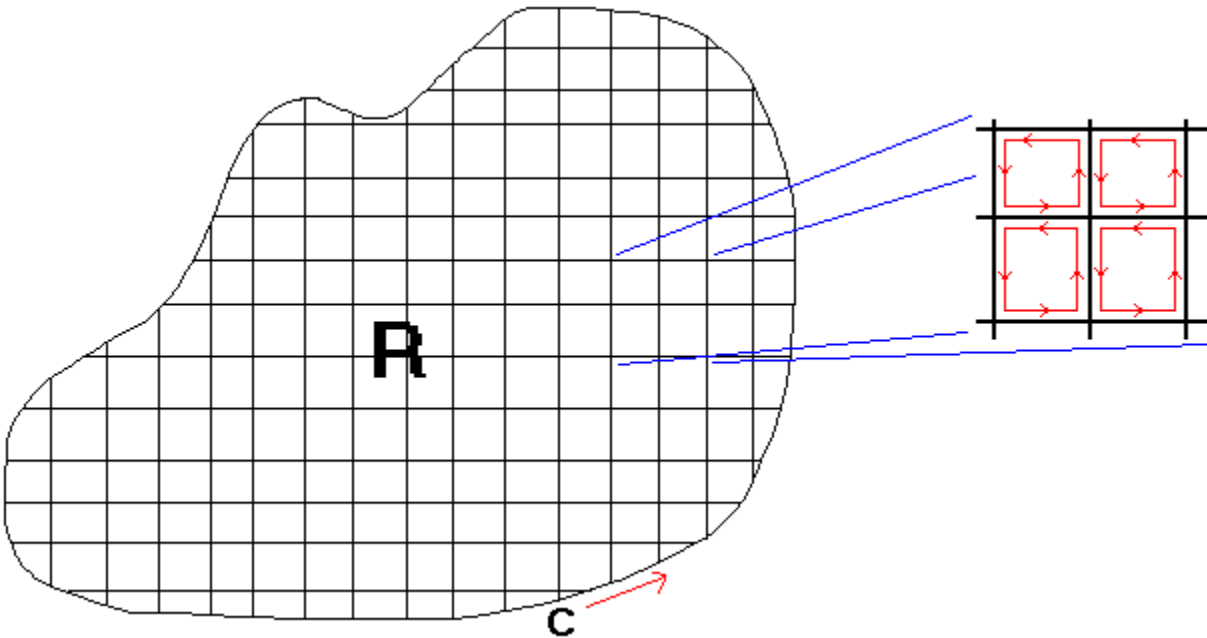


Why Green's Theorem is true

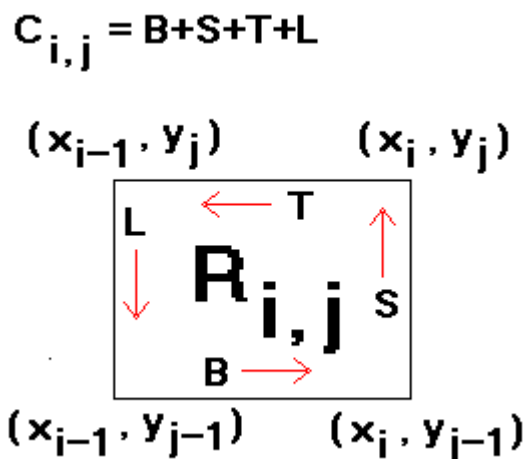
Green's Theorem: Let C be a simple closed curve enclosing a region R in the xy -plane, oriented so that R is on the left as one faces along C . Let $\vec{F} = F_1\vec{i} + F_2\vec{j}$ be a smooth vector field defined on all of C and R . Then:

$$\int_C \vec{F} \cdot d\vec{r} = \int_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA .$$



Note first that both integrals in the definition are the sums of the corresponding integrals over a partition of R into subregions, mostly rectangles with some odd “leftovers” along the edges, next to C : For the integral over R , this is clear from the definition of double integral. For the line integral over C , note that each the line integrals along internal edges is cancelled by the line integral in the adjoining subregion, because in going around that rectangle the edge is oriented in the opposite direction. As we refine the partition, the double integral over the “leftover” regions becomes negligible, because the total area of these regions becomes negligible; but the total line integral remains exactly the line integral over C .

So we only need to check Green's Theorem holds on one of the small rectangles $R_{i,j}$. We can break up the boundary $C_{i,j}$ into the bottom B , the right side S , the top T and the left side L :



Then we can parameterize T , for example, by $x = -t$, $y = y_j$, $-x_{i-1} \leq t \leq -x_i$, and have, using the substitution $x = -t$:

$$\int_T \vec{F} \cdot d\vec{r} = \int_{-x_{i-1}}^{-x_i} F_1(-t, y_j) dt = - \int_{x_{i-1}}^{x_i} F_1(x, y_j) dx .$$

Similarly (or more simply) for B , S , and L , so we get:

$$\begin{aligned}
 \int_{R_{i,j}} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA &= \int_{x_{i-1}}^{x_i} \int_{y_{j-1}}^{y_j} \frac{\partial F_2}{\partial x} dy dx - \int_{x_{i-1}}^{x_i} \int_{y_{j-1}}^{y_j} \frac{\partial F_1}{\partial y} dy dx \\
 &= \int_{y_{j-1}}^{y_j} \int_{x_{i-1}}^{x_i} \frac{\partial F_2}{\partial x} dx dy - \int_{x_{i-1}}^{x_i} \int_{y_{j-1}}^{y_j} \frac{\partial F_1}{\partial y} dy dx \\
 &= \int_{y_{j-1}}^{y_j} (F_2(x_i, y) - F_2(x_{i-1}, y)) dy \\
 &\quad - \int_{x_{i-1}}^{x_i} (F_1(x, y_j) - F_1(x, y_{j-1})) dx \\
 &= \int_{y_{j-1}}^{y_j} F_2(x_i, y) dy - \int_{y_{j-1}}^{y_j} F_2(x_{i-1}, y) dy \\
 &\quad - \int_{x_{i-1}}^{x_i} F_1(x, y_j) dx + \int_{x_{i-1}}^{x_i} F_1(x, y_{j-1}) dx \\
 &= \int_S \vec{F} \cdot d\vec{r} + \int_L \vec{F} \cdot d\vec{r} + \int_T \vec{F} \cdot d\vec{r} + \int_B \vec{F} \cdot d\vec{r} \\
 &= \int_{C_{i,j}} \vec{F} \cdot d\vec{r}
 \end{aligned}$$

So Green's Theorem works on the small rectangle $R_{i,j}$ and therefore over all of R .