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6. The only critical point of this function occurs at the origin, and it is a saddle point (see page 48); so the extrema are on the boundary of the square. Now on the edges $x = \pm 1$, the surface forms a parabola ($z = 1 - y^2$) concave down, with maximum value 1 at $(\pm 1, 0)$ and minimum value 0 at $(\pm 1, \pm 1)$. On the edges $y = \pm 1$, the surface forms a parabola ($z = x^2 - 1$) concave up, with maximum value 0 at $(\pm 1, \pm 1)$ and minimum value -1 at $(0, \pm 1)$. So the global extrema of the function on the square are minimum -1 at $(0, \pm 1)$ and maximum 1 at $(\pm 1, 0)$.

8. (a) The negative coefficients for p_1 in the expression for q_1 and for p_2 in q_2 are hardly surprising: As the price of an item goes down, demand for it will go up. The negative coefficients on p_2 in the expression for q_1 and vice versa are more interesting (though the interpretation above is still accurate). Apparently the items are not competing but cooperating; if one gets cheaper, the other becomes more in demand. Perhaps one such pair of items might be VCRs and cable television: a home with many channels available would be more likely to want to tape concurrent programs, and a home with a VCR would be more likely to want cable TV, to have more programs worth taping.
(b) The manufacturer wants to maximize

$$\begin{aligned} R &= p_1 q_1 + p_2 q_2 = p_1(150 - 2p_1 - p_2) + p_2(200 - p_1 - 3p_2) \\ &= 150p_1 + 200p_2 - 2p_1^2 - 2p_1p_2 - 3p_2^2 \ . \end{aligned}$$

We have $\partial R/\partial p_1 = 150 - 4p_1 - 2p_2$ and $\partial R/\partial p_2 = 200 - 2p_1 - 6p_2$; setting them equal to 0, we get the system of linear equations $2p_1 + p_2 = 75$ and $p_1 + 3p_2 = 100$. It is easy to solve this system (in one of several different ways; for example, solve the second for p_1 — $p_1 = -3p_2 + 100$ — and substitute into the first) to get $p_1 = p_2 = 25$ dollars (if that is the monetary unit), and then $R = 4375$ dollars. (And this really is a maximum: The function is quadratic, so it is one of the catalogue on page 48, and because $D = (-4)(-6) - (-2)^2 > 0$ and $\partial^2 R/\partial p_1^2 = -4 > 0$, it is a paraboloid opening upwards.)

9. Profit is given by revenue minus cost, so:

$$\begin{aligned} P &= pq - C_1 - C - 2 = (60 - 0.04(q_1 + q_2))(q_1 + q_2) - (8.5 + 0.03q_1^2) - (5.2 + 0.04q_2)^2 \\ &= -13.7 + 60q_1 + 60q_2 - 0.07q_1^2 - 0.08q_1q_2 - 0.08q_2^2 \end{aligned}$$

We have $\partial P/\partial q_1 = 60 - 0.14q_1 - 0.08q_2$ and $\partial P/\partial q_2 = 60 - 0.08q_1 - 0.16q_2$; setting them equal to 0, we get the system of linear equations $7q_1 + 4q_2 = 3000$ and $q_1 + 2q_2 = 750$. This system has solution $q_1 = 300$, $q_2 = 225$ (in whatever units the product is measured). (And this really is a maximum: It is quadratic, and $D = (-0.14)(-0.16) - (-0.08)^2 > 0$ and $\partial^2 P/\partial q_1^2 = -0.14 < 0$, so it is a paraboloid opening downwards.)

11. $\partial R/\partial t = -10t - 6h + 400$, and $\partial R/\partial h = -6t - 6h + 300$. Setting them equal to 0 gives $5t + 3h = 200$ and $t + h = 50$, so $t = 25°$C and $h = 25$ percent humidity. The function is quadratic, $D = (-10)(-6) - (-6)^2 > 0$, and $\partial^2 R/\partial t^2 = -10 < 0$, so these conditions give a maximum range for missile control.

14. From $w\ell h = 512$ we get $h = 512/(\ell w)$, so the total cost in cents is $C = 2(2\ell w) + 1(2wh + 2\ell h) = 4\ell w + 1024(\ell + w)/(\ell w)$. Now

$$\begin{aligned} C_\ell &= 4w + 1024\frac{\ell w - (\ell + w)w}{(\ell w)^2} = 4w - 1024\frac{1}{\ell^2} \\ C_w &= 4\ell + 1024\frac{\ell w - (\ell + w)\ell}{(\ell w)^2} = 4\ell - 1024\frac{1}{w^2} \end{aligned}$$

Setting both equal to 0 gives $4w\ell^2 = 1024 = 4\ell w^2$, so $\ell = w$; and hence $\ell^3 = 256$; so $w = \ell = 256^{1/3} \approx 6.35$ cm and $h = 512/256^{2/3} \approx 12.70$ cm. This is the only candidate for minimizing the cost; letting $\ell = w$ get arbitrarily large would make the first term of C arbitrarily large; and letting them approach 0 would make the second term arbitrarily large. So these critical point dimensions yield a global minimum cost.

15. To get the greatest volume, we clearly want the height of the package to be as large as it legally can be for a given width and length, so $h = 135 - \ell - w$ and we want to maximize $V = \ell w h = \ell w(135 - \ell - w)$: $V_\ell = \ell w(-1) + w(135 - \ell - w)$ and $V_w = \ell w(-1) + \ell(135 - \ell - w)$. Setting the partials equal to 0 (and dismissing $w = 0$ and $\ell = 0$ because we want a maximum volume) gives $\ell = 135 - \ell - w = w$, and hence $135 = 3w$, so $w = \ell = 45$ cm (and the height is $135 - (45) - (45) = 45$ cm also). Because this is the only critical point of a function that can only sensibly have a maximum, it must give the maximum volume.