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2. Let us begin by noting that $x^2 + 2y^2 = 44$ is closed and bounded (an ellipse), so we know that both maxima and minima of f exist on it. To find them, we need to solve the system of equations

$$3 = \lambda(2x) \ , \qquad -2 = \lambda(4y) \ , \qquad x^2 + 2y^2 = 44 \ .$$

Solving for x in the first and y in the second, and substituting into the third, we get $(3/(2\lambda))^2 + 2(-2/(4\lambda))^2 = 44$, or $1 = 16\lambda^2$, so $x = \pm 3/(2/4) = \pm 6$ and $y = \mp 2/(4/4) = \mp 2$, so the critical points are $(6, -2)$ and $(-6, 2)$. Now $f(6, -2) = 22$ and $f(-6, 2) = -22$, so $(6, -2)$ is a maximum and $(-6, 2)$ is a minimum.

3. Note first that $x^2 - y^2 = 1$ is a hyperbola; it isn't bounded, so there may fail to be either or both a maximum or a minimum value of f on it. In fact, as x and y both increase on one branch (the one asymptotic to $y = x$ for positive x-values), we see that f gets arbitrarily large; so we will not find a global maximum for f on this curve. But everywhere on the curve we have $f = (1 + y^2) + y$, which is parabola opening upward; as y gets large (positive or negative) on all the branches of the hyperbola, f increases without bound, so there will be a minimum point for f somewhere on it.
To find it, we need to solve the system

$$2x = \lambda(2x) \ , \qquad 1 = \lambda(-2y) \ , \qquad x^2 - y^2 = 1 \ .$$

From the first equation, either $x = 0$ or $\lambda = 1$. From the third equation, $x = 0$ gives $-y^2 = 1$, so there are no solutions. So $\lambda = 1$, and from the second equation $y = -\frac{1}{2}$, so from the third equation $x = \pm\sqrt{1 + (-\frac{1}{2})^2} = \pm\sqrt{5}/2$. The points $(-\frac{1}{2}, \pm\frac{\sqrt{5}}{2})$ both make $f = \frac{3}{4}$, so they are both the desired minima.

10. The region $x^2 + y^2 + z^2 = 1$ is closed and bounded (a sphere) so f does have maximum and minimum values on it. To find them, we need to solve the system

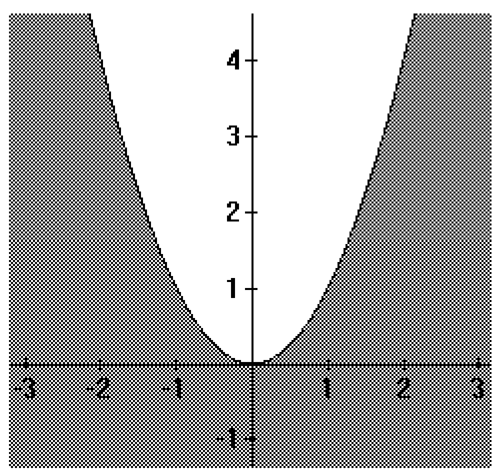
$$2x = \lambda(2x) \ , \qquad -2 = \lambda(2y) \ , \qquad 4z = \lambda(2z) \ , \qquad x^2 + y^2 + z^2 = 1 \ .$$

From the first equation, either $\lambda = 1$ or $x = 0$; and from the third either $\lambda = 2$ or $z = 0$. From the second equation $y = -1/\lambda$. So substituting into the fourth equation gives one of the following:

$\lambda = 1,\ y = -1,\ z = 0$	$\lambda = 2,\ y = -\frac{1}{2},\ x = 0$	$\lambda \neq 1, 2,\ x = z = 0$
$x^2 + (-1)^2 + 0^2 = 1$	$0^2 + (-\frac{1}{2})^2 + z^2 = 1$	$0^2 + y^2 + 0^2 = 1$
$x = 0$	$z = \pm\sqrt{3}/2$	$y = 1$ (not -1: $\lambda \neq 1$)

So the critical points are $(0, \pm 1, 0)$ and $(0, -\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$. We have $f(0, -1, 0) = 2$, $f(0, 1, 0) = -2$ and $f(0, -\frac{1}{2}, \pm\frac{\sqrt{3}}{2}) = \frac{5}{2}$, so $(0, -\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$ are maxima and (0,1,0) is a minimum.

14. The region on which we are to extremize f is shaded:



It is unbounded, so f may not attain maximum and/or minimum values on it. In fact, if we fix $y = 0$

and let x get large (which it may do in the region; the positive x-axis is in it), we see that f can get arbitrarily large, so f has no maximum. Similarly, if we let y get large (positive) and take $x = \sqrt{y}$ (along one edge of the region), then $f = y - y^2$ (in the yf-plane, a parabola opening downward), which can take on arbitrarily large negative values; so f also has no global minimum on the region. So all we might find are local extrema.

To do so, first we consider unconstrained critical points: $2x = 0$ and $-2y = 0$ when $x = y = 0$, on the edge of the legal region anyway. So we look for critical points of f subject to $x^2 - y = 0$: Solve the system

$$2x = \lambda(2x) \ , \qquad -2y = \lambda(-1) \ , \qquad x^2 = y$$

to get either $x = 0$ (and then $y = 0$ also) or $\lambda = 1$ (and then $y = 1/2$ and $x = \pm 1/\sqrt{2}$). We have $f(0,0) = 0$ and $f(\pm 1/\sqrt{2}, 1/2) = 1/4$, so $(0,0)$ is a local minimum on $x^2 = y$ and $(\pm 1/\sqrt{2}, 1/2)$ are local maxima on that curve. Are they at least local extrema for the whole region $x^2 \geq y$? I.e., for example, if we move into the region $x^2 > y$ a small distance from (0,0), does f increase from 0? The answer is no: If we fix $x = 0$ and take a small negative y (which is in the region), then $f < 0$. And if we move into that region away from $(\pm 1/\sqrt{2}, 1/2)$, does f decrease? This answer is no: If we fix $y = 1/2$ but take x slightly larger than $1/\sqrt{2}$ or slightly smaller than $-1/\sqrt{2}$, then f is larger than $1/4$. So the points we found are not even local extrema on the region.

17. The region $x^2 + y^2 \leq 1$ is closed and bounded, so f will have maximum and minimum values on it somewhere. First we find the unconstrained critical points: $3x^2 = 0$ and $-2y = 0$ when $x = y = 0$, and that is a point properly inside the legal region. [Note that $D = (6x)(-2) - 0^2 = -12x$ is 0 at (0,0), so the second derivative test doesn't help classify this extremum. It turns out to be a saddle point, but we know there are global extrema to be found, so we'll treat it as a candidate and test it with the others.] Turning to the edge of the region, we extremize f subject to $x^2 + y^2 = 1$: Solve the system

$$3x^2 = \lambda(2x) \ , \qquad -2y = \lambda(2y) \ , \qquad x^2 + y^2 = 1 \ .$$

From the first equation, either $x = 0$ or $\lambda = \frac{3}{2}x$, and and from the second either $y = 0$ or $\lambda = -1$. So substituting into the third equation gives one of the following:

		$x \neq 0$ and $y \neq 0$
$x = 0$	$y = 0$	$-1 = \lambda = \frac{3}{2}x$
$0^2 + y^2 = 1$	$x^2 + 0^2 = 1$	$x = -\frac{2}{3}$
$y = \pm 1$	$x = \pm 1$	$(-\frac{2}{3})^2 + y^2 = 1$
		$y = \pm\frac{\sqrt{5}}{3}$

Now $f(0,0) = 0$, $f(0,\pm 1) = -1$, $f(1,0) = 1$, $f(-1,0) = -1$ and $f(-2/3, \pm\sqrt{5}/3) = -23/27$, so (1,0) is a (global, on the region) maximum and $(-1,0)$ and $(0,\pm 1)$ are (global) minima.

18. We want to minimize $C = 20x + 10y + 5z$ subject to $1200 = 20x^{1/2}y^{1/4}z^{2/5}$, or more simply $x^{1/2}y^{1/4}z^{2/5} = 60$. So we need to solve the system

$$20 = \lambda(\frac{1}{2}x^{-1/2}y^{1/4}z^{2/5}) \ , \qquad 10 = \lambda(\frac{1}{4}x^{1/2}y^{-3/4}z^{2/5}) \ ,$$
$$5 = \lambda(\frac{2}{5}x^{1/2}y^{1/4}z^{-3/5}) \ , \qquad x^{1/2}y^{1/4}z^{2/5} = 60 \ .$$

Multiplying the first of these by $2x$, the second by $4y$ and the third by $\frac{5}{2}z$ shows that $40x$, $40y$ and $\frac{25}{2}z$ are all equal to $\lambda x^{1/2}y^{1/4}z^{2/5}$, so they are equal to each other: $x = y = \frac{5}{16}z$. Substituting into the fourth equation gives

$$\begin{aligned}
\left(\frac{5}{16}z\right)^{1/2}\left(\frac{5}{16}z\right)^{1/4} z^{2/5} &= 60 \\
\left(\frac{5}{16}\right)^{3/4} z^{23/20} &= 60 \\
z &= \left(60\left(\frac{16}{5}\right)^{3/4}\right)^{20/23} \approx 75.1
\end{aligned}$$

and $x = y \approx \frac{5}{16}(75.1) \approx 23.5$.

23. Denote the radius, height and surface area of the cylinder by r, h and A respectively. We want to minimize $A = 2\pi r^2 + 2\pi rh$ subject to $\pi r^2 h = 100$, so we need to solve

$$4\pi r + 2\pi h = \lambda(2\pi rh)\ , \qquad 2\pi r = \lambda(\pi r^2)\ , \qquad \pi r^2 h = 100\ ;$$

or more simply

$$2r + h = \lambda rh\ , \qquad r(2 - \lambda r) = 0\ , \qquad \pi r^2 h = 100\ .$$

From the second equation, because $r = 0$ is impossible (the cylinder must have a positive radius to have a volume), we see that $\lambda r = 2$, and substituting this into (the simpler version of) the first equation gives $2r + h = 2h$, or $h = 2r$. Substituting this into the third equation gives $2\pi r^3 = 100$, or $r = (50/\pi)^{1/3}$; and then $h = 2(50/\pi)^{1/3}$.