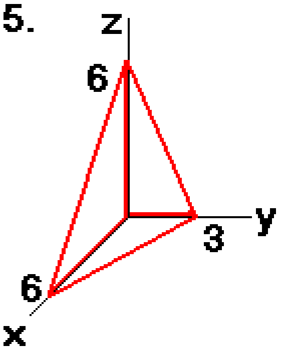
Problems 15.3, Page 238

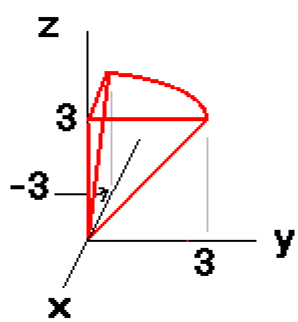
2.

$$\int_0^\pi \int_0^\pi \int_0^\pi \sin x \cos(y+z)\, dz\, dy\, dx = \int_0^\pi \int_0^\pi [\sin x \sin(y+z)]_{z=0}^{z=\pi}\, dy\, dx$$
$$= \int_0^\pi \int_0^\pi \sin x(\sin(y+\pi) - \sin y)\, dy\, dx = \int_0^\pi \int_0^\pi \sin x(-\sin y - \sin y)\, dy\, dx$$
$$= -2\int_0^\pi \int_0^\pi \sin x \sin y\, dy\, dx = -2\left(\int_0^\pi \sin x\, dx\right)\left(\int_0^\pi \sin y\, dy\right)$$
$$= -2\left([-\cos x]_0^\pi\right)^2 = -2(-1-1)^2 = -8$$

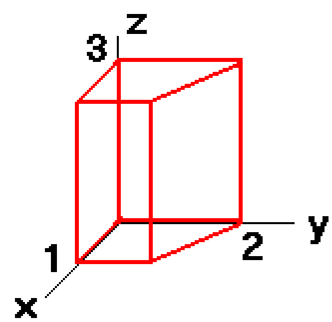
5.



8.



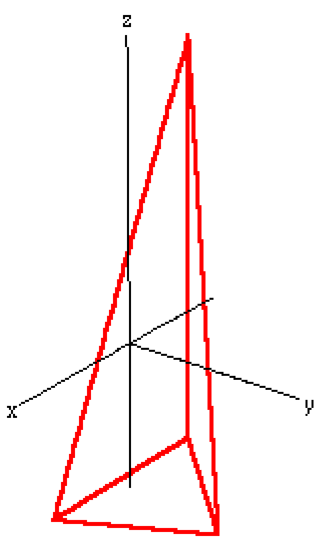
10.



7. Does not make sense: The upper limit on the (middle) y-integral is z, which is the variable of integration in the innermost integral.

9. Does not make sense: The upper limit on the (middle) x-interval involves an x.

12. Note that this problem is very similar to Chapter 15, Section 2, problem 21; the difference is that the base is in the plane $z = -6$, which also changes the limits on the variables. Here is the graph:



The slanted plane $2x + y + z = 4$ meets the plane $z = -6$ in the line $2x + y = 10$; so these two planes and the plane $y - x = 4$ meet at $(2, 6, -6)$ (by solving $2x + y = 10$ and $y - x = 4$ simultaneously). Also the lines $2x + y = 10$ and $y - x = 4$ in the plane $z = -6$ meet the plane $y = 0$ at $(5, 0, -6)$ and $(-4, 0, -6)$ respectively. So:

$$\begin{aligned}
\text{Volume} &= \int_0^6 \int_{y-4}^{(10-y)/2} \int_{-6}^{4-2x-y} 1\, dz\, dx\, dy = \int_0^6 \int_{y-4}^{(10-y)/2} (10 - 2x - y)\, dx\, dy \\
&= \int_0^6 \left[(10-y)x - x^2\right]_{y-4}^{(10-y)/2} dy \\
&= \int_0^6 ((10-y)\frac{1}{2}(10-y) - \frac{1}{4}(10-y)^2 - ((10-y)(y-4) - (y-4)^2))\, dy \\
&= \int_0^6 (\frac{1}{2}(10-y) - (y-4))^2\, dy = \int_0^6 (9 - \frac{3}{2}y)^2\, dy = \int_0^6 (81 - 27y + \frac{9}{4}y^2)\, dy \\
&= \left[81y - \frac{27}{2}y^2 + \frac{3}{4}y^3\right]_0^6 = 162
\end{aligned}$$

13. The slanted plane, which can be written $z = 6 - 2x - 3y$, meets the x-axis at $(3, 0, 0)$ and the xy-plane in $(x/3) + (y/2) = 1$, i.e., $y = 2(1 - x/3)$. So:

$$\begin{aligned}
\text{Mass} &= \int_0^3 \int_0^{2(1-x/3)} \int_0^{6-2x-3y} (x+y)\, dz\, dy\, dx = \int_0^3 \int_0^{2(1-x/3)} (x+y)(6 - 2x - 3y)\, dy\, dx \\
&= \int_0^3 \int_0^{2(1-x/3)} (6x + 6y - 2x^2 - 5xy - 3y^2)\, dy\, dx \\
&= \int_0^3 \left[(6x - 2x^2)y + (3 - \frac{5}{2}x)y^2 - y^3\right]_0^{2(1-x/3)} dx \\
&= \int_0^3 ((6x - 2x^2)(2 - \frac{2}{3}x) + (3 - \frac{5}{2}x)(2 - \frac{2}{3}x)^2 - (2 - \frac{2}{3}x)^3)\, dx \\
&= \int_0^3 (12x - 4x^2 - 4x^2 + \frac{4}{3}x^3 + 12 - 8x + \frac{4}{3}x^2 - 10x + \frac{20}{3}x^2 - \frac{10}{9}x^3 \\
&\qquad -8 + 8x - \frac{8}{3}x^2 + \frac{8}{27}x^3)\, dx \\
&= \int_0^3 \left(4 + 2x - \frac{8}{3}x^2 + \frac{14}{27}x^3\right) dx = \left[4x + x^2 - \frac{8}{9}x^3 + \frac{7}{54}x^4\right]_0^3 = \frac{15}{2}
\end{aligned}$$

14. The volume of the region in question is $2^3 = 8$, so:

$$\begin{aligned}
\text{Average} &= \frac{1}{8}\int_0^2 \int_0^2 \int_0^2 (x^2 + y^2 + z^2)\, dz\, dy\, dx = \frac{1}{8}\int_0^2 \int_0^2 \left[(x^2 + y^2)z + \frac{1}{3}z^3\right]_0^2 dy\, dx \\
&= \frac{1}{8}\int_0^2 \int_0^2 \left(2x^2 + 2y^2 + \frac{8}{3}\right) dy\, dx = \frac{1}{8}\int_0^2 \left[2x^2y + \frac{2}{3}y^3 + \frac{8}{3}y\right]_0^2 dx \\
&= \frac{1}{8}\int_0^2 \left(4x^2 + \frac{32}{3}\right) dx = \frac{1}{8}\left[\frac{4}{3}x^3 + \frac{32}{3}x\right]_0^2 = \frac{1}{8}\left(\frac{32}{3} + \frac{64}{3}\right) = 4
\end{aligned}$$

16. First we need the mass of the solid. (I evaluated this integral, but it's fine if you just wrote it down and called it M — or whatever — in the other formulas.)

$$\begin{aligned}
\text{Mass} = M &= \int_0^1 \int_0^1 \int_0^{x+y+1} 1\, dz\, dy\, dx = \int_0^1 \int_0^1 (x + y + 1)\, dy\, dx \\
&= \int_0^1 \left[(xy + \frac{1}{2}y^2 + y\right]_0^1 dx = \int_0^1 (x + \frac{3}{2})\, dx = \left[\frac{1}{2}x^2 + \frac{3}{2}x\right]_0^1 = 2\ .
\end{aligned}$$

So the desired coordinates are:

$$\overline{x} = \frac{1}{2}\int_0^1\int_0^1\int_0^{x+y+1} x\,dz\,dy\,dx \qquad \overline{y} = \frac{1}{2}\int_0^1\int_0^1\int_0^{x+y+1} y\,dz\,dy\,dx$$

$$\overline{z} = \frac{1}{2}\int_0^1\int_0^1\int_0^{x+y+1} z\,dz\,dy\,dx$$

Though you weren't asked to evaluate them, these coordinates turn out to be $(13/24, 13/24, 25/24)$.