Problems 15.6, Page 254

2. Using the substitution $u = r^2$ (so that $du = 2r\,dr$ and as r goes from 0 to 1, so does u), we get:

$$\begin{aligned}\int_{-1}^{3}\int_{0}^{2\pi}\int_{0}^{1}\sin(r^2)\,r\,dr\,d\theta\,dz &= \left(\int_{-1}^{3}dz\right)\left(\int_{0}^{2\pi}d\theta\right)\left(\int_{0}^{1}r\sin(r^2)\,dr\right)\\ &= 4(2\pi)\left[-\frac{1}{2}\cos(u)\right]_0^1 = 4\pi(1-\cos 1)\ .\end{aligned}$$

3.

$$\int_{0}^{2\pi}\int_{\pi/2}^{\pi}\int_{0}^{5}\frac{1}{\rho}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \left(\int_{0}^{2\pi}d\theta\right)\left(\int_{\pi/2}^{\pi}\sin\phi\,d\phi\right)\left(\int_{0}^{5}\rho\,d\rho\right) = 2\pi\left[-\cos\phi\right]_{\pi/2}^{\pi}\left(\frac{25}{2}\right) = 25\pi\ .$$

6. This is a job for cylindrical coordinates: Put the positive x-axis along the edge marked 2, the positive y-axis along the lower edge in the back left, and the positive z-axis along the edge marked 4. Then the integral is

$$\int_{0}^{4}\int_{0}^{\pi/2}\int_{0}^{2}\delta\cdot r\,dr\,d\theta\,dz\ .$$

7. This looks to me like a reasonable case for spherical coordinates: Put the positive z-axis through the vertical axis of symmetry of the solid and the x- and y-axes in the horizontal plane through the point at the bottom of the solid (perpendicular to each other, of course). Then the integral is

$$\int_{0}^{2\pi}\int_{0}^{\pi/6}\int_{0}^{3}\delta\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta\ .$$

8. More spherical coordinates: Put the origin at the center of the hollow hemisphere, with the positive x-and z-axes roughly in the directions of the arrows marked 3 and 2 respectively, and the positive y-axis aimed into the hollowed-out part of the solid. Then the integral is

$$\int_{0}^{\pi}\int_{0}^{\pi}\int_{2}^{3}\delta\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta\ .$$

9. Finally, rectangular coordinates: Put the origin at the lower left corner of the face toward us, the positive x- and z-axes along the lower and left edges of the face, and the positive y-axis along the lower edge of the thin face in the back to our left. Then the integral is

$$\int_{0}^{3}\int_{0}^{1}\int_{0}^{5}\delta\,dz\,dy\,dx\ .$$

11. This looks like a job for spherical coordinates: The region over which we integrate is the forward half of a sphere centered at the origin, of radius 1 (forward, that is, if we are as usual looking almost straight down the x-axis at the yz-plane); and the integrand is just $1/\rho$. So we can rewrite the integral as

$$\int_{-\pi/2}^{\pi/2}\int_{0}^{\pi}\int_{0}^{1}\frac{1}{\rho}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \left(\int_{-\pi/2}^{\pi/2}d\theta\right)\left(\int_{0}^{\pi}\sin\phi\,d\phi\right)\left(\int_{0}^{1}\rho\,d\rho\right) = \pi\left(\left[-\cos\phi\right]_0^{\pi}\right)\frac{1}{2} = \pi\ .$$

12. The region over which we integrate is the solid cylinder centered on the z-axis, between $z = 0$ (the xy-plane) and $z = 1$, of radius 1; and the integrand is just $1/r$. So we can rewrite the integral in cylindrical coordinates:

$$\int_{0}^{1}\int_{0}^{2\pi}\int_{0}^{1}\frac{1}{r}r\,dr\,d\theta\,dz = \left(\int_{0}^{1}dz\right)\left(\int_{0}^{2\pi}d\theta\right)\left(\int_{0}^{1}dr\right) = 1(2\pi)1 = 2\pi\ .$$

17. The following integral breaks up as the difference of two integrals, the first of which is just 3 times the volume of the spherical cloud, i.e., $3(4/3)\pi(3)^3 = 108\pi$. But we still need to evaluate the other:

$$\begin{aligned}
\text{Mass} &= \int_0^{2\pi}\int_0^{\pi}\int_0^{3}(3-\rho)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = 108\pi - \int_0^{2\pi}\int_0^{\pi}\int_0^{3}\rho^3\sin\phi\,d\rho\,d\phi\,d\theta \\
&= 108\pi - \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi}\sin\phi\,d\phi\right)\left(\int_0^{3}\rho^3\,d\rho\right) \\
&= 108\pi - 2\pi\left([-\cos\phi]_0^{\pi}\right)\left(\left[\frac{1}{4}r^4\right]_0^3\right) = 108\pi - 2\pi(2)\frac{81}{4} = 27\pi\ .
\end{aligned}$$

We weren't told the units of the density function D, but assuming it was kilograms per cubic kilometer, the result is 27π kg. [I see by the answer book that the authors don't break it into two integrals, and it still isn't too bad.]

22.

$$(a)\ \ \text{Mass} = \int_0^{2\pi}\int_0^{1}\int_r^{1} z^2 r\,dz\,dr\,d\theta = 2\pi\int_0^1 r\frac{1}{3}(1-r^3)\,dr = \frac{2}{3}\pi\left[\frac{1}{2}r^2 - \frac{1}{5}r^5\right]_0^1 = \frac{\pi}{5}$$

$$\begin{aligned}
(b)\ \ \overline{z} &= \frac{5}{\pi}\int_0^{2\pi}\int_0^{1}\int_r^{1} z^3 r\,dz\,dr\,d\theta = \frac{5}{\pi}\left(2\pi\int_0^1 r\frac{1}{4}(1-r^4)\,dr\right) \\
&= \frac{5}{\pi}\left(\frac{1}{2}\pi\left[\frac{1}{2}r^2 - \frac{1}{6}r^6\right]_0^1\right) = \frac{5}{\pi}\left(\frac{\pi}{6}\right) = \frac{5}{6}
\end{aligned}$$