Problems 18.1, Page 348

Solutions to Exercises 26 and 28 are included, though they weren't assigned, because they show applications of line integrals.

2. Negative: The curve moves "against the current" of the vector field.

3. Positive: The curve moves with the vector field.

4. Zero: The curve first moves with the field, then against it, and the two halves seem to match.

5. The integral along C_3 is likely to be negative, along C_1 zero, and along C_2 positive, so:

$$\int_{C_3} \vec{F} \cdot d\vec{r} < \int_{C_1} \vec{F} \cdot d\vec{r} < \int_{C_2} \vec{F} \cdot d\vec{r}$$

6. The total integral is likely to be zero: The integrals along C_1 and C_3 are probably 0, and the integrals along C_2 and C_4 cancel out.

7. The total integral is likely to be positive: The integrals along C_2 and C_4 are probably 0, but the integral along C_3 is positive and larger in absolute value than the integral along C_1, because C_3 is longer and the field vectors along it are larger.

12. The field is constantly $2\vec{\jmath}$ along the "curve", which has length 5, in the direction of $\vec{\jmath}$, so the line integral is $2 \cdot 5 = 10$.

15. If we let r denote the length of $\vec{r}$, i.e., the distance from the origin, then the length of the field vector at each point along the "curve" is r, and its direction is in the direction of the curve; so the line integral is

$$\int_{2\sqrt{2}}^{6\sqrt{2}} r\, dr = \left[\frac{1}{2}r^2\right]_{2\sqrt{2}}^{6\sqrt{2}} = \frac{1}{2}(72 - 8) = 32 \ .$$

24. Suppose that there were two curves, say C_1 and C_2, from P to Q. Then the curve $C_1 + (-C_2)$, as defined on page 348, is a closed curve, so by the hypothesis the integral of the vector field $\vec{F}$ along it is 0. Thus, using the properties on page 348:

$$0 = \int_{C_1+(-C_2)} \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{-C_2} \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r}$$

and so

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r} \ .$$

26. At each point of its trip, both the gravitiational force and the increment in position are in line with the particle's position vector $\vec{r}$, in opposite directions; so their dot product is just the negative of the product of their lengths:

$$-\frac{GMm}{r^3}\vec{r} \cdot d\vec{r} = -\frac{GMm}{r^3} r\, dr = -GMm\frac{1}{r^2}\, dr \ .$$

Integrating this, we get

$$-GMm \int_{8000}^{10000} \frac{1}{r^2}\, dr = GMm \left[r^{-1}\right]_{8000}^{10000} = GMm \left(\frac{1}{10000} - \frac{1}{8000}\right) = -\frac{GMm}{40000} \ ,$$

in whatever units we are supposed to be using.

28. Unfortunately, this problem defines r as a constant, the radius of the circle C, so it would be confusing to use dr as the length of an infinitesimal piece of the circle C. So let $\vec{s}$ denote the position vector of a

point on the circle C relative to its center on the wire. Then $d\vec{s}$ is tangent to C, so it is parallel to $\vec{B}$ at each point. Thus, at each point the dot product $\vec{B} \cdot d\vec{s}$ is the product of the lengths, $\|\vec{B}\|\, ds$, where as usual ds means the length of $d\vec{s}$. The integral adds up, over all the pieces, to the constant $\|\vec{B}\|$ times the length $2\pi r$ of C; so we get

$$\|\vec{B}\|(2\pi r) = \int_C \vec{B} \cdot d\vec{s} = kI \ ,$$

and the equation in the problem follows by dividing both ends by $2\pi r$.