In today's 11:20 class, Karen Kelley asked two questions that I couldn't answer immediately. Here are some answers:

The first question was from the text, Exercise 16.29: Must every ring with a prime number of elements be commutative? We saw immediately that if $(R,+,*)$ is a ring for which $|R|$ is a prime number, then $(R,+)$ is a cyclic group. But the question is whether the multiplication $*$ has to be commutative. I claim that it does, because $*$ is distributive over $+$. To see this, let a be a generator of the additive cyclic group R. I contend that the entire multiplication table for $*$ is determined by the choice of the product $a*a$, and that it turns out to be commutative: Every element of R has the form $na = a+a+\cdots+a$ (n terms) for some positive integer n (from 1 up to the prime order of R, to be exact, but we don't use that here). Now

$$\begin{aligned}(na)*(ma) &= (a+a+\cdots+a)*(a+a+\cdots+a) \qquad (n \text{ and } m \text{ terms})\\ &= a*a+a*a+\cdots+a*a \qquad (mn \text{ terms})\\ &= nm(a*a) = mn(a*a) = (ma)*(na)\ ,\end{aligned}$$

where we have used the fact that multiplication of integers <u>is</u> commutative. Therefore $*$ is commutative, i.e., R is a commutative ring.

The second question may be in the text, but I can't find it. The question was: Find an example of a ring R and an ideal I in it, and an ideal J of I that is not an ideal in R. The example that I've come up with seems to me much harder than it should be, but at least it works. Let $R = \mathbb{R}[x, \mathbb{Q}^+ \cup \{0\}]$, the "semigroup ring of $\mathbb{Q}^+ \cup \{0\}$ over $\mathbb{R}$", i.e., all polynomials in x with coefficients in $\mathbb{R}$, but allowing any nonnegative <u>rational</u> number as an exponent. We get polynomials that look like, for example, $2 + x^{1/2} + 5x^2 - 3x^{10/3}$. Now let:

$$S = \{q \in \mathbb{Q} : q \geq 1\} \quad \text{and} \quad T = \{1\} \cup \{q \in \mathbb{Q} : q \geq 2\}\ .$$

The set of elements with exponents in S is an ideal of R — the product of a polynomial with lowest-degree term at least x^1 and any other polynomial in R is another with lowest-degree term at least x^1. And the set J of polynomials with exponents in T is an ideal in I — in fact, the product of <u>any</u> two elements of I is in J, because its lowest degree term is at least x^2. But J is not an ideal in R: $x^{1/2} \in R$ and $x^1 \in J$, but $x^{1/2}x^1 = x^{3/2} \notin J$. There <u>must</u> be a simpler example.

P.S.: Here is another example; whether it qualifies as simpler is up to you: Let R be the set of polynomials in x with terms ax^n where a is a real number and $n = 2r + 3s$ for some nonnegative integers r, s — so the exponents can be any element of $\mathbb{N} \cup \{0\}$ except 1. (In the terms of the last example, this R is the semigroup ring of the "numerical semigroup generated by 2 and 3".) Now let I be the set of elements of R with zero coefficients in the x^0 (i.e., constant) and x^2 terms, and let J be the set of elements of R with zero coefficients in the x^0, x^2, and x^5 terms. The product of any two elements of I has lowest exponent at least 6, so it is in J; so J captures multiplication in I and hence is an ideal there. But J is not an ideal in R, because $x^3 \in J$ and $x^2 \in R$ but $x^5 \notin J$.