The Fibonacci sequence \( f_1, f_2, f_3, \ldots \) is defined by setting \( f_1 = 1, f_2 = 1 \), and for \( n \geq 3 \), \( f_n = f_{n-1} + f_{n-2} \) (so \( f_3 = 2, f_4 = 3, f_5 = 5 \), etc.). Prove that, for every positive integer \( n \), the sum of the squares of the first \( n \) terms in the Fibonacci sequence, \( \sum_{j=1}^{n} f_j^2 \), equals \( f_n f_{n+1} \).

2. Which of the following are groups? For those which are not, tell why not.
   (a) \( (\mathbb{Z}^+, \text{exponentiation}) \)
   (b) \( (\mathbb{R} - \{0\}, \text{multiplication}) \)
   (c) \( (\mathbb{Z}^+, +) \)
   (d) \( (GL(2, \mathbb{R}), \text{addition of matrices}) \)

3. Let \( S \) be a set with an associative operation \( \ast \) and an identity \( e \).
   (a) Prove that an element of \( S \) has at most one inverse.
   (b) Prove that if \( a, b \) in \( S \) have inverses, then \( a \ast b \) also has an inverse.

4. The operation table to the right defines a group.
   (a) Is this group abelian? Prove your answer.
   (b) Is this group cyclic? Prove your answer.

5. (a) In each of the following groups, list the elements of the cyclic group generated by the given element:
   (i) \( (\mathbb{R}^+, \cdot) \), 5
   (ii) \( (\mathbb{Z}_{10}, +) \), 6
   (iii) \( (\mathcal{P}(\{1, 2, 3\}), \Delta) \), \{1, 2\}
   (b) Give the order of 4, regarded as an element of each of the following groups:
   (i) \( (\mathbb{Z}_{10}, \oplus) \)
   (ii) \( (\mathbb{Q}^+, \cdot) \)
   (iii) \( (\mathbb{Z}_{15}, \oplus) \)

6. The “least common multiple” \( \text{lcm}(a, b) \) of two nonzero integers \( a, b \) is the smallest positive integer which is divisible (evenly) by both \( a \) and \( b \). Let \( x, y \) be elements of an abelian group, and \( m = \text{lcm}(o(x), o(y)) \).
   (a) Prove that \( o(xy) \) divides \( m \).
   (b) Prove that, if \( \langle x \rangle \cap \langle y \rangle = \{e\} \), then \( o(xy) = m \). (Hint: If \( (xy)^n = e \), then \( x^n = y^{-n} \).

7. Let \( x, g \) be elements of a group, with \( o(x) = 10 \). Why is it impossible that \( gxg^{-1} = x^5 \)?
Solutions to Exam I

1. For \( n = 1 \): \[ \sum_{j=1}^{1} j = 1^2 = 1 \]. Assume that for a particular \( n \), \[ \sum_{j=1}^{n} j^2 = f_n f_{n+1} \]. Then

\[ \sum_{j=1}^{n+1} j^2 = \sum_{j=1}^{n} j^2 + f_{n+1}^2 = f_n f_{n+1} + f_{n+1}^2 = f_{n+1}(f_n + f_{n+1}) = f_{n+1} f_{n+2} \].

It follows that the desired equation holds for all positive integers \( n \).

2. (a) Not a group: Exponentiation is not associative.
   (b) A group.
   (c) Not a group: There is no additive identity in \( \mathbb{Z}^+ \).
   (d) Not a group: The sum of two invertible matrices may not be invertible, so addition of matrices is not an operation on \( \text{GL}(2, \mathbb{R}) \).

3. (a) Suppose \( x \) in \( S \) has inverses \( y \) and \( z \); then \( y = ye = y(xz) = (yx)z = ez = z \), so the “two” inverses are the same element.
   (b) \((a * b) * (b^{-1} * a^{-1}) = a * (b * b^{-1}) * a^{-1} = a * e * a^{-1} = e \), and similarly \((b^{-1} * a^{-1}) * (a * b) = e \), so \( b^{-1} * a^{-1} \) is an inverse for \( a * b \).

4. (a) No: \( on = r \) but \( no = q \).
   (b) No: Cyclic groups are abelian, and we have just seen that this group is not abelian.

5. (a) (i) \{\ldots, \frac{1}{25}, \frac{1}{5}, 1, 5, 25, 125, \ldots\},  (ii) \{6, 2, 8, 4, 0\},  (iii) \{1, 2\}, \emptyset
   (b) (i) 5,  (ii) \infty,  (iii) 15

6. (a) Because the group is abelian, \((xy)^m = x^my^m\), and because \( m \) is a multiple of both \( o(x) \) and \( o(y) \), both \( x^m \) and \( y^m \) are \( e \), so \( (xy)^m = ee = e \), so the order of \( xy \) divides \( m \).
   (b) Suppose \((xy)^n = e\); then as noted in the hint, \( x^n = y^{-n}\), so this element, in both \( \langle x \rangle \) and \( \langle y \rangle \), is \( e \): \( x^n = e \) and \( y^n = e \). Thus, \( n \) is a common multiple of both \( o(x) \) and \( o(y) \), and so it is greater than or equal to \( \text{lcm}(o(x), o(y)) = m \). So \( m \) is the smallest power of \( xy \) that is \( e \), i.e., \( o(xy) = m \).

7. The conjugates of \( x \) all have the same order as \( x \) itself, i.e., 10, but \( x^5 \) has order 2, so it cannot be a conjugate of \( x \).