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Math 320 — Exam I

Make sure your reasoning is clear, in most cases in English sentences, with symbols used only for abbreviation and then used correctly.

1. The Fibonacci sequence $f_1, f_2, f_3, \ldots$ is defined by setting $f_1 = 1$, $f_2 = 1$, and for $n \geq 3$, $f_n = f_{n-1} + f_{n-2}$ (so $f_3 = 2$, $f_4 = 3$, $f_5 = 5$, etc.). Prove that, for every positive integer n, the sum of the squares of the first n terms in the Fibonacci sequence, $\sum_{j=1}^{n} f_j^2$, equals $f_n f_{n+1}$.

2. Which of the following are groups? For those which are not, tell why not.
   (a) ($\mathbb{Z}^+$, exponentiation)
   (b) ($\mathbb{R} - \{0\}$,multiplication)
   (c) $(\mathbb{Z}^+, +)$
   (d) ($GL(2, \mathbb{R})$,addition of matrices)

3. Let S be a set with an associative operation $*$ and an identity e.
   (a) Prove that an element of S has at most one inverse.
   (b) Prove that if a, b in S have inverses, then $a * b$ also has an inverse.

4. The operation table to the right defines a group.
   (a) Is this group abelian? Prove your answer.
   (b) Is this group cyclic? Prove your answer.

	m	n	o	p	q	r
m	m	n	o	p	q	r
n	n	p	q	m	r	o
o	o	r	m	q	p	m
p	p	m	r	n	o	q
q	q	o	n	r	m	p
r	r	q	p	o	n	m

5. (a) In each of the following groups, list the elements of the cyclic group generated by the given element:

   (i) $(\mathbb{R}^+, \cdot)$, 5 (ii) $(\mathbb{Z}_{10}, +)$, 6 (iii) $(\mathcal{P}(\{1,2,3\}), \triangle)$, $\{1,2\}$

   (b) Give the order of 4, regarded as an element of each of the following groups:

   ((i) $(Z_{10}, \oplus)$ (ii) $(\mathbb{Q}^+, \cdot)$ (iii) $(\mathbb{Z}_{15}, \oplus)$

6. The "least common multiple" lcm(a, b) of two nonzero integers a, b is the smallest positive integer which is divisible (evenly) by both a and b. Let x, y be elements of an abelian group, and $m = \text{lcm}(o(x), o(y))$.
   (a) Prove that $o(xy)$ divides m.
   (b) Prove that, if $\langle x \rangle \cap \langle y \rangle = \{e\}$, then $o(xy) = m$. (Hint: If $(xy)^n = e$, then $x^n = y^{-n}$.)

7. Let x, g be elements of a group, with $o(x) = 10$. Why is it impossible that $gxg^{-1} = x^5$?

Solutions to Exam I

1. For $n = 1$: $\sum_{j=1}^{1} 1^2 = 1(1)$. Assume that for a particular n, $\sum_{j=1}^{n} f_j^2 = f_n f_{n+1}$. Then

$$\sum_{j=1}^{n+1} f_j^2 = \sum_{j=1}^{n} f_j^2 + f_{n+1}^2 = f_n f_{n+1} + f_{n+1}^2 = f_{n+1}(f_n + f_{n+1}) = f_{n+1} f_{n+2} \ .$$

   It follows that the desired equation holds for all positive integers n.

2. (a) Not a group: Exponentiation is not associative.
   (b) A group.
   (c) Not a group: There is no additive identity in $\mathbb{Z}^+$.
   (d) Not a group: The sum of two invertible matrices may not be invertible, so addition of matrices is not an operation on $GL(2, \mathbb{R})$.

3. (a) Suppose x in S has inverses y and z; then $y = ye = y(xz) = (yx)z = ez = z$, so the "two" inverses are the same element.
   (b) $(a * b) * (b^{-1} * a^{-1}) = a * (b * b^{-1}) * a^{-1}) = a * e * a^{-1} == a * a^{-1} = e$, and similarly $(b^{-1} * a^{-1}) * (a * b) = e$, so $b^{-1} * a^{-1}$ is an inverse for $a * b$.

4. (a) No: $on = r$ but $no = q$.
   (b) No: Cyclic groups are abelian, and we have just seen that this group is not abelian.

5. (a) (i) $\{\ldots, \frac{1}{25}, \frac{1}{5}, 1, 5, 25, 125, \ldots\}$, (ii) $\{6, 2, 8, 4, 0\}$, (iii) $\{\{1, 2\}, \emptyset\}$
   (b) (i) 5, (ii) ∞, (iii) 15

6. (a) Because the group is abelian, $(xy)^m = x^m y^m$, and because m is a multiple of both $o(x)$ and $o(y)$, both x^m and y^m are e, so $(xy)^m = ee = e$, so the order of xy divides m.
   (b) Suppose $(xy)^n = e$; then as noted in the hint, $x^n = y^{-n}$, so this element, in both $\langle x \rangle$ and $\langle y \rangle$, is e: $x^n = e$ and $y^n = e$. Thus, n is a common multiple of both $o(x)$ and $o(y)$, and so it is greater than or equal to $\text{lcm}(o(x), o(y)) = m$. So m is the smallest power of xy that is e, i.e., $o(xy) = m$.

7. The conjugates of x all have the same order as x itself, i.e., 10, but x^5 has order 2, so it cannot be a conjugate of x.