March 22, 2011

Math 320 — Exam II

Make sure your reasoning is clear, in most cases in English sentences, with symbols used only for abbreviation and then used correctly. (Possible total points 75.)

1. *(20 points)* (a) Which of the following groups are cyclic? (Maybe all, maybe some, maybe none.)

   (i) $\mathbb{Z}_4 \times \mathbb{Z}_5$ (ii) $\mathbb{Z}_{12} \times \mathbb{Z}_{15}$ (iii) $\mathbb{Z}_8 \times \mathbb{Z}_5 \times \mathbb{Z}_{12}$

   (b) Find a nonabelian group of order (cardinality) 30.

   (c) Fix an element (x, y) of a direct product of groups $G \times H$. Prove that the centralizer $Z((x, y))$ of (x, y) is the direct product $Z(x) \times Z(y)$ of the centralizers of x in G and y in H.

   (d) If A, B are subgroups of G, H respectively, then $A \times B$ is always a subgroup of $G \times H$. Find a subgroup K of $\mathbb{Z}_3 \times \mathbb{Z}_3$ that does not have the form $A \times B$ where A is a subgroup of the first $\mathbb{Z}_3$ and B is a subgroup of the second $\mathbb{Z}_3$.

2. *(10 points)* (a) Find the disjoint cycle decomposition of the element of $\mathcal{S}_{10}$ given by

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 4 & 3 & 2 & 8 & 7 & 6 & 5 & 10 & 9 & 1 \end{pmatrix}$$

   (b) Is the element of $\mathcal{S}_{10}$ in (a) an odd or even permutation? Explain.

3. *(24 points)* Which of the following are equivalence relations? For those that are, describe the equivalence classes. For those that are not, show one property to equivalence relation that each does not have.

   (a) On $\mathbb{R}$, define the relation $\mathcal{C}$ by: $x \,\mathcal{C}\, y$ iff $|x - y| \leq 1$ (i.e., iff x, y are no more than one unit apart).

   (b) For the real number x, let $\lfloor x \rfloor$ denote the *floor function* of x, i.e., the greatest integer less than or equal to x. (So, for example, we have $\lfloor 2 \rfloor = 2$, $\lfloor \pi \rfloor = 3$, and $\lfloor -2.5 \rfloor = -3$.) Define the relation $\mathcal{F}$ on $\mathbb{R}$ by $x \,\mathcal{F}\, y$ iff $\lfloor x \rfloor = \lfloor y \rfloor$.

   (c) On the family $\mathcal{P}(X)$ of subsets of a nonempty set X, take the familiar relation "is a subset of".

4. *(10 points)* Let G be a finite group, and let H, K be subgroups of G with $K \subseteq H$. Use Lagrange's theorem to prove that

$$[G : K] = [G : H][H : K] \ .$$

5. *(11 points)* Let H be a subgroup of the symmetric group of degree n, $\mathcal{S}_n$, and as usual let $\mathcal{A}_n$ denote the alternating group of degree n, i.e., the set of even permutations in $\mathcal{S}_n$. Prove that either $H \subseteq \mathcal{A}_n$ or else exactly half the elements of H are in $\mathcal{A}_n$. (Hint: Suppose the first alternative doesn't hold, and take $q \in H - \mathcal{A}_n$. Use q to define functions between the sets $H \cap \mathcal{A}_n$ and $H - \mathcal{A}_n$ that are inverse to each other.)

Solutions to Exam II

1. (a) (i) Cyclic, because $\gcd(4,5) = 1$. (ii) Not cyclic, because $\gcd(12,15) \neq 3$. (iii) Not cyclic, because $\gcd(8,12) \neq 1$.

   (b) One answer is $\mathcal{S}_3 \times \mathbb{Z}_5$.

   (c)

$$\begin{aligned} (a,b) \in Z((x,y)) \quad &\iff \quad (a,b)(x,y) = (x,y)(a,b) \quad \iff \quad (ax,by) = (xa,yb) \\ &\iff \quad ax = xa \text{ and } by = yb \quad \iff \quad a \in Z(x) \text{ and } b \in Z(y) \\ &\iff \quad (a,b) \in Z(x) \times Z(y) \ . \end{aligned}$$

   (d) One such subgroup is $K = \langle(1,1)\rangle = \{(1,1),(2,2),(0,0)\}$. It is nontrivial, so if it were of the form $A \times B$, then at least one of A, B would be nontrivial; so K would include an element of the form $(a,0)$ where $a \neq 0$, because $0 \in B$, or of the form $(0,b)$ where $b \neq 0$, because $0 \in A$; but there are no such elements.

2. (a) $(1,4,8,10)(2,3)(5,7)$

   (b) This element is odd, because it is the product of an odd number of even-length (and hence odd as permutations) cycles.

3. (a) This is not an equivalence relation because it is is not transitive: $1 \,\mathcal{C}\, 2$ and $2 \,\mathcal{C}\, 3$, but not $1 \,\mathcal{C}\, 3$.

   (b) This is an equivalence relation, with classes are the half-open intervals $[n, n+1)$ where $n \in \mathbb{Z}$.

   (c) This is not an equivalence relation because it is not symmetric: $\emptyset \subseteq X$ but not $X \subseteq \emptyset$.

4. From $|G| = [G:H]|H|$, $|G| = [G:K]|K|$ and $|H| = [H:K]|K|$, we get

$$[G:K]|K| = |G| = [G:H]|H| = [G:H][H:K]|K| \ ,$$

   so if we cancel $|K|$ from both ends, we get the result.

5. As the hint suggests, suppose $H \not\subseteq \mathcal{A}_n$, and pick an odd element q from H. We can define

$$\varphi : H \cap \mathcal{A}_n \to H - \mathcal{A}_n : x \to xq \qquad \text{and} \qquad \psi : H - \mathcal{A}_n \to H \cap \mathcal{A}_n : y \to yq^{-1}$$

   — q^{-1} is also an odd element in H, so multiplying even elements of H by q gives odd elements in H, and multiplying odd elements of H by q^{-1} gives even elements in H. Now for all x in $H \cap \mathcal{A}_n$ and for all y in $H - \mathcal{A}_n$,

$$(\psi \circ \varphi)(x) = xqq^{-1} = x \qquad \text{and} \qquad (\varphi \circ \psi)(y) = yq^{-1}q = y \ .$$

   Thus, $\psi \circ \varphi$ is the identity function on $H \cap \mathcal{A}_n$, and $\varphi \circ \psi$ is the identity function on $H - \mathcal{A}_n$. In other words, φ and ψ are inverses of each other and hence are both bijections. So $|H \cap \mathcal{A}_n| = |H - \mathcal{A}_n|$, i.e., exactly half the elements of H are in $\mathcal{A}_n$.