4/9/81

Math 320 — Exam II

Make sure your reasoning is clear, in most cases in English sentences with symbols used only for abbreviation and then used correctly.

1. Which of the following subsets are subgroups? For each which is not, give one reason why not.

   (a) $\{0\}$ in $\mathbb{Z}$ (b) $\mathbb{Z}_5$ in $\mathbb{Z}$ (c) $\{e, g, fg\}$ in D_4 (d) $\mathbb{R}^+$ in $\mathbb{R}$

2. Prove that $SL(n, \mathbb{R})$, the set of $n \times n$ matrices with real entries and determinant 1, is a subgroup of $GL(n, \mathbb{R})$. You may use the fact from linear algebra that $\det(AB) = (\det A)(\det B)$.

3. Let $G = \langle x \rangle$ be a cyclic group of order pq where p, q are distinct primes. Draw the subgroup lattice of G.

4. Find groups of order 16 which are:
   (a) cyclic (b) abelian but not cyclic (c) nonabelian

5. Let $f : S \to T$ be a function. A "leftinverse" for f is a function $g : T \to S$ for which $g \circ f$ is the identity function on S.

   (a) Prove that if f has a left inverse, then f is 1-1.
   (b) Let $S = \{a, b\}$, $T = \{x, y, z\}$, and $f = \{(a, x)(b, y)\}$. Find two left inverses g_1, g_2 for f.
   (c) Define (by analogy) what is meant by a "right inverse" h for f.
   (d) Prove that if f has a right inverse, then f is onto T.

6. Recall that D_n, the dihedral group of the n-gon, and A_n, the alternating group of degree n, are subgroups of S_n, the symmetric group of degree n.

   (a) For which n is the element f of D_n also in A_n?
   (b) For which n is the element g of D_n also in A_n? (Give your answer in terms of the remainder of n on division by 4.)
   (c) If f, g are in A_n, what is $D_n \cap A_n$?

7. On the group $\mathbb{Z}_{10}$, define relations as follows: For a, b in $\mathbb{Z}_{10}$, aR_1b iff $a \oplus b = 0$ and aR_2b iff $\langle a \rangle = \langle b \rangle$.

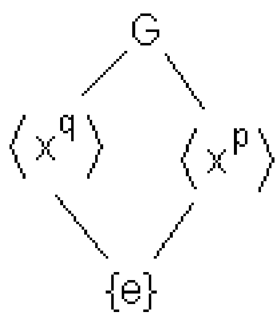
   (a) Which of R_1 and R_2 are equivalence relations? (Both, maybe.)
   (b) For each that is an equivalence relation, find the equivalence classes in $\mathbb{Z}_{10}$.
   (c) For each that is not an equivalence relation, give examples from $\mathbb{Z}_{10}$ for each defining property of equivalence relation that fails.

Solutions to Exam II

1. (a) A subgroup. (b) Not a subgoup: $\mathbb{Z}_5$ is a group, but its operation (addition mod 5) is different from the usual operation of addition on $\mathbb{Z}$. (c) Not a subgroup: Not closed under the operation, because $g(fg) = (gf)g = f^3gg = f^3$. (d) No additive identity or inverses.

2. The identity matrix I has determinant 1, so it is in $SL(n, \mathbb{R})$. If $A, B \in SL(n, \mathbb{R})$, i.e., $\det(A) = \det(B) = 1$, then $\det(AB) = \det(A)\det(B) = 1(1) = 1$, so $AB \in SL(n, \mathbb{R})$. If $A \in SL(n, \mathbb{R})$, then $\det(A^{-1}) = \det(A)\det(A^{-1}) = \det(AA^{-1}) = \det(I) = 1$, so $A^{-1} \in SL(n, \mathbb{R})$. Therefore, $SL(n, \mathbb{R})$ is a subgroup of $GL(n, \mathbb{R})$.

3.



4. Here are some possible answers: (a) $\mathbb{Z}_{16}$ (b) $\mathbb{Z}_4 \times \mathbb{Z}_4$ (c) $D_4 \times \mathbb{Z}_2$.

5. (a) As the problem says, denote a left inverse of f by g. Now assume s_1, s_2 are elements of S for which $f(s_1) = f(s_2)$. Then $s_1 = i(s_1) = (g \circ f)(s_1) = g(f(s_1)) = g(f(s_2)) = (g \circ f)(s_2) = i(s_2) = s_2$.
(b) $g_1 = \{(x, a), (y, b), (z, a)\}$, $g_2 = \{(x, a), (y, b), (z, b)\}$
(c) A right inverse of f is a function $h : T \to S$ for which $f \circ h$ is the identity function on T.
(d) Let t be an element of T. Then $t = i(t) = (f \circ h)(t) = f(h(t))$, so t is in the image of f.

6. (a) f is an n-cycle, which we know is the composition of $n - 1$ transpositions. So f is in A_n if $n - 1$ is even, i.e., if n is odd.
(b) If n is even, then g is the composition of $(n-2)/2$ transpositions, so g is even if $(n-2)/2$ is even, i.e., the remainder on division by 4 is 2; and g is odd if that remainder is 0. If n is odd, then g is the composition of $(n - 1)/2$ transpositions; so g is even if $(n - 1)/2$ is even, i.e., the remainder on division of n by 4 is 1, and odd if the remainder is 3. So g is in A_n if the remainder when n is divided by 4 is 1 or 2, and not in A_n if the remainder is 0 or 3.
(c) Because all elements of D_n are products of f and g, if they are in A_n, then $D_n \subseteq A_n$, so $D_n \cap A_n = D_n$.

7. (a) R_1 is not an equivalence relation, but R_2 is one.
(b) For R_2, the equivalence classes are $\{0\}$, $\{1, 3, 7, 9\}$, $\{2, 4, 6, 8\}$, $\{5\}$.
(c) R_1 isn't reflexive: $1 \oplus 1 = 2 \neq 0$. It isn't transitive: $1 \oplus 9 = 0$ and $9 \oplus 1 = 0$, but $1 \oplus 1 \neq 0$. But it is symmetric, because $\oplus$ is commutative.