April 19, 2011

Math 320 — Exam III

Make sure your reasoning is clear. (Possible total points 75.)

1. *(28 points)* Which of the following functions are homomorphisms? For those that are, which are monomorphisms, epimorphisms or isomorphisms? For those that are homomorphisms, what are the kernel, the image, and the two groups that are isomorphic according to the Fundamental Theorem of Group Homomorphisms?

   (a) $(\mathbb{R}, +) \to (\mathbb{R}^+, \cdot) : x \mapsto 2^x$

   (b) $(\mathbb{Z}_{12}, \oplus) \to (\mathbb{Z}_{12}, \oplus) : x \mapsto 3x$

   (c) $\det : GL(2, \mathbb{R}) \to \mathbb{R} - \{0\}$

   (d) For a group G and a fixed nonidentity element a of G, $G \to G : g \mapsto ag$

2. *(12 points)* It is a fact that a finite abelian group of order m is isomorphic to the direct product of groups $\mathbb{Z}_q$ where the q's are powers of primes that multiply to m. (So, "up to isomorphism", the only abelian groups of order 4 are $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$.)

   (a) Using this fact, find all the possible abelian groups of order 18, up to isomorphism.

   (b) Give a nonabelian group of order 18.

3. *(15 points)* Let H be a subgroup of a group G. In an exercise, you proved that $N(H) = \{g \in G : gHg^{-1} = H\}$ is a subgroup of G containing H (the largest subgroup in which H is normal).

   (a) Prove that the conjugate subgroups xHx^{-1} and yHy^{-1} are equal if and only if the left cosets $xN(H)$ and $yN(H)$ are equal.

   (b) Assume $|G|$ is finite. Use (a) to conclude that the number of conjugates of H in G is a factor of $[G : H]$.

4. *(20 points)* Let G be a group.

   (a) Let H, K be subgroups of G. Assume that:

      (i) for all h in H and k in K, we have $hk = kh$ (so that $HK = \{hk : h \in H,\ k \in K\}$ is a subgroup of G, but you don't need to prove that),

      (ii) $H \cap K = \{e\}$, and

      (iii) $HK = G$.

      Prove that $H \times K$ is isomorphic to G. (In this case, G is called the "internal direct product" of H and K.)

   (b) Suppose G is abelian, let m be a positive integer, and let $H = \{h \in G : o(h)|m\}$. Prove that H is a subgroup of G.

   (c) Now suppose G is finite abelian, and that $|G| = mn$ where $\gcd(m, n) = 1$. Let H be as in (b) and $K = \{k \in G : o(k)|n\}$ (also a subgroup of G, by (b)).

      (i) Prove that $H \cap K = \{e\}$.

      (ii) Prove that $HK = G$. (Hint: For g in G, write $o(g) = rs$ where $r|m$ and $s|n$, and $1 = ar + bs$. Set $h = g^{bs}$ and $k = g^{ar}$.)

Solutions to Exam III

1. (a) Isomorphism: Kernel $\{0\}$, image $\mathbb{R}^+$, $\mathbb{R} \cong \mathbb{R}^+$.
   (b) Homomorphism: Kernel $\langle 4 \rangle$, image $\langle 3 \rangle$, $\mathbb{Z}_{12}/\langle 4 \rangle \cong 3\mathbb{Z}_{12}$.
   (c) Epimorphism: Kernel $SL(2, \mathbb{R})$, image $\mathbb{R} - \{0\}$, $GL(2\mathbb{R})/SL(2, \mathbb{R}) \cong \mathbb{R} - \{0\}$.
   (d) Not a homomorphism: It is one-to-one function from G onto itself, the image of a under the monomorphism $\psi : G \to \mathcal{S}(G)$ used in Cayley's Theorem, but it does not respect the operation on G.

2. (a) $\mathbb{Z}_2 \times \mathbb{Z}_9$ and $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$.
   (b) D_9 (or $S_3 \times \mathbb{Z}_3$).

3. (a)

$$\begin{aligned} xHx^{-1} = yHy^{-1} &\iff y^{-1}xHx^{-1}y = H \iff (y^{-1}x)H(y^{-1}x)^{-1} = H \\ &\iff y^{-1}x \in N(H) \iff y^{-1}xN(H) = N(H) \\ &\iff xN(H) = yN(H) \end{aligned}$$

   (b) By (a), the number of conjugates of H in G is the number of left cosets of $N(H)$, i.e., $[G : N(H)]$. And because $N(H)$ is a subgroup between H and G, we have $[G : H] = [G : N(H)][N(H) : H]$.

4. (a) Define $\varphi : H \times K \to G : (h, k) \mapsto hk$. Then φ is a homomorphism because G is abelian:

$$\varphi((h_1, k_1)(h_2, k_2)) = \varphi((h_1h_2, k_1k_2)) = h_1h_2k_1k_2 = h_1k_1h_2k_2 = \varphi((h_1, k_1))\varphi((h_2, k_2)) \ .$$

   And $(h, k) \in \ker(\varphi)$ iff $hk = e$, i.e., $h = k^{-1}$, so h and k must both be in $H \cap K$, i.e., $h = k = e$; thus, φ is one-to-one. Finally, φ has image $HK = G$, so φ is an epimorphism and hence an isomorphism.
   (b) $o(e) = 1|m$, so $e \in H$, so $H \neq \emptyset$. If $x, y \in H$, then $o(x), o(y)|m$, so $x^m = e = y^m$, so, using the fact that G is abelian, $(xy)^m = x^m y^m = ee = e$, so $o(xy)|m$, so $xy \in H$. If $x \in H$, then $o(x^{-1}) = o(x)|m$, so $x^{-1} \in H$. So H is a subgroup of G.
   (c) (i) If $g \in H \cap K$, then $o(g)$ divides both m, n, so it divides their gcd, 1; so $g = e$. (ii) Take any g in G; then $o(g)||G| = mn$, and because m, n are relatively prime, they have no prime factors in common. So we can factor $o(g)$, which divides mn, as rs where the prime factors of r divide m and those of s divide n. Then $\gcd(r, s) = 1$ also, so we can write $1 = ar + bs$ where $a, b \in \mathbb{Z}$. Because r is a factor of m, $o(g) = rs$ is a factor of bsm, so $(g^{bs})^m = e$, i.e., $g^{bs} \in H$; and similarly $g^{ar} \in K$. Thus, $g = (g^{bs})(g^{ar}) \in HK$.