4/30/81

Math 320 — Exam III

Make sure your reasoning is clear, in most cases in English sentences with symbols used only for abbreviation and then used correctly. Proofs need not be long.

1. In a group G of 34 elements, what are the possible orders of subgroups of G? What are the possible orders of elements of G? Is there a single group of 34 elements which has elements of all possible orders?

2. Give reasons why three of the following subgroups are normal in the corresponding groups. And show that the fourth is not normal.

   (a) $\langle 5\rangle$ in $\mathbb{Z}_{30}$ (b) $\langle f\rangle$ in D_5 (c) $\langle(1,2)\rangle$ in S_3 (d) $SL(3,\mathbb{R})$ in $GL(3,\mathbb{R})$

3. Let $\varphi : G \to H$ be an epimorphism of groups and N be a normal subgroup of G. Prove that $\varphi(N)$ is normal in H.

4. Sketch a proof for each of the following (i.e., define functions and state what must be checked).

   (a) $(\mathbb{R} - \{0\})/\{\pm 1\} \cong \mathbb{R}^+$

   (b) $\mathbb{Z}/\langle 4\rangle$ is isomorphic to a subgroup of $\mathbb{C} - \{0\}$. (Hint: Send 1 to i.)

5. Consider the group D_{17} (which is too large to test all cases of anything).

   (a) What are $Z(f^k)$ (for any $k = 1, \ldots, 16$) and $Z(g)$? (See Problem 1. Hlnt: Since $g \in Z(g)$, 2 divides $|Z(g)|$.)

   (b) How many conjugates does f have? How many does g have?

   (c) What are the conjugates of f ?

   (d) Write the class equation of D_{17} — in its number form, not its set form.

Solutions to Exam III

1. Possible orders of subgroups, the divisors of 34, i.e. 1, 2, 17 and 34. Possible orders of elements, the same. The cyclic group $\mathbb{Z}_{34}$ has exactly one subgroup of each possible order (and that is the only one, up to isomorphism, because no other 34-element group has an element of order 34).

2. (a) $\mathbb{Z}_{30}$ is abelian, so all its subgroups are normal.
   (b) $[D_5 : \langle f \rangle] = 2$.
   (c) Not normal: $(1,2,3,4,5)(1,2)(1,2,3,4,5)^{-1} = (1,3,4,5)(1,5,4,3,2) = (2,3) \notin \langle(1,2)\rangle$.
   (d) $SL(3,\mathbb{R})$ is the kernel of the determinant, which is a group homomorphism $GL(3,\mathbb{R}) \to \mathbb{R} - \{0\}$.

3. I don't read the question as requiring that we show $\varphi(N)$ is a subgroup, but just in case, let's throw that in: Because $e_G \in N$, we have $\varphi(e_G) \in \varphi(N)$, so $\varphi(N) \neq \emptyset$. If $x, y \in \varphi(N)$, then $x = \varphi(a)$ and $y \in \varphi(b)$ for some $a, b \in N$, and $ab \in N$, so $xy = \varphi(a)\varphi(b) = \varphi(ab) \in \varphi(N)$; and $a^{-1} \in N$, so $x^{-1} = \varphi(a)^{-1} = \varphi(a^{-1}) \in \varphi(N)$. Thus, $\varphi(N)$ is a subgroup. To show normality, take $x \in \varphi(N)$ and $z \in H$. Then $x = \varphi(a)$ for some a in N, as before, and $z = \varphi(c)$ for some c in G because φ is onto H. Now $cac^{-1} \in N$ because $N \lhd G$, so $zxz^{-1} = \varphi(c)\varphi(a)\varphi(c)^{-1} = \varphi(cac^{-1}) \in \varphi(N)$. Therefore, $\varphi(N) \lhd H$.

4. (a) Define $|\cdot| : \mathbb{R} - \{0\} \to \mathbb{R}^+$ by $r \mapsto |r|$. Then $|\cdot|$ is a epimorphism of multiplicative groups, and the kernel is $\{1, -1\}$, so by the Fundamental Theorem of Group Homomorphisms, $(\mathbb{R} - \{0\})/\{\pm 1\} \cong \mathbb{R}^+$.
   (b) Define $\varphi : \mathbb{Z} \to \mathbb{C} - \{0\} : n \mapsto i^n$. Then φ is a group homomorphism with kernel $\langle 4 \rangle$, because i^4 is the smallest positive power of i that equals 1; and its image is $\langle i \rangle$, so by the Fundamental Theorem of Group Homomorphisms, $\mathbb{Z}/\langle 4 \rangle \cong \langle i \rangle$, which is a subgroup of $\mathbb{C} - \{0\}$.

5. (a) Because $f^{-k}g = gf^k$, and $f^{-k} \neq f^k$ for any $k = 1, \ldots, 16$ because 17 is odd, we see that $g \notin Z(f^k)$, so $Z(f^k)$ is not all of D_{17}. But all the powers of f commute with each other, so $\langle f \rangle \subseteq Z(f^k)$. Because $\langle f \rangle$ is a subgroup of D_{17} of index 2, a prime, there are no subgroups properly between it and D_{17}, so $\langle f \rangle = Z(f^k)$. Similarly, $Z(g)$ contains at least $\langle g \rangle$, which has index 17, another prime, and $Z(g)$ is a proper subgroup of D_{17}, so $Z(g) = \langle g \rangle$.
   (b) f has $[D_{17} : Z(f)] = [D_{17} : \langle f \rangle] = 34/17 = 2$ conjugates, and g has $[D_{17} : Z(g)] = [D_{17} : \langle g \rangle] = 34/2 = 17$ conjugates.
   (c) f is one of its own conjugates, and the other one must be the result of conjugating by one of the elements in the other left coset $g\langle f \rangle$ of its centralizer: $gfg^{-1} = f^{-1}gg = f^{16}$.
   (d) The 16 powers of f (other than e) are conjugate in inverse pairs, and g has 17 conjugates; so a set of representatives of the conjugacy classes with more than one element is $\{f, f^2, \ldots, f^8, g\}$, and the class equation is

$$\begin{aligned} |D_{17}| &= |Z(D_{17})| + 8[D_{17} : \langle f \rangle] + [D_{17} : \langle g \rangle] \\ 34 &= 1 + 8(2) + 17 \end{aligned}$$