5/19/81

Math 320 — Final Exam

Clear explanations, preferably in English sentences, increase chances for partial credit.

1. (a) Let S be a set with an associative operation $*$, and let x be an element of S. Prove by induction that, for any positive integer n, x^n commutes with x. (You may use, as the definition of x^n, the product $x * \cdots * x$, with n factors, if $n > 1$, or x if $n = 1$.)
   (b) Prove that this <u>fails</u> if $*$ is not associative. (Hint: Try $S = \mathbb{Z}$, $* =$ subtraction.)

2. Give the cyclic subgroups generated by each of the following elements, and the orders of these elements.

   (a) i in $\mathbb{C} - \{0\}$ (b) 3 in $\mathbb{Z}$ (c) $(3, 1)$ in $\mathbb{Z}_6 \times \mathbb{Z}_4$ (d) $\{1\}$ in $\mathcal{P}(\{1, 2\})$

3. (a) What are the possible orders of subgroups of D_5?
   (b) Draw the lattice of subgroups of D_5.
   (c) Which subgroups of D_5 are normal? Which are not? Prove your answers for $\langle f \rangle$ and $\langle g \rangle$.

4. Let G be a subgroup of S_n. On the set $\{1, 2, \ldots, n\}$, define the relation R on G by xRy iff there is an element g of G such that $g(x) = y$.

   (a) Prove that R is an equivalence relation.
   (b) Pick a particular x in $\{l, 2, \ldots, n\}$, and let $G_x = \{g \in G \ : \ g(x) = x\}$. Prove that G_x is a subgroup of G.
   (c) Prove that, for g_1, g_2 in G, $g_1(x) = g_2(x)$ iff $g_1 G_x = g_2 G_x$.
   (d) Conclude from (c) that the number of images of x under elements of G is a factor of $|G|$.

5. (a) Prove that there is no epimorphism from $\mathbb{Z}_{10}$ onto $\mathbb{Z}_4$.
   (b) Prove that there is no epimorphism from $\mathbb{Z}_{12}$ onto V.

6. In the ring $\mathbb{Z}_{12}$ find: (a) a zerodivisor other than 0; (b) a nilpotent other than 0; (c) an idempotent other than 0 and 1; and (d) a unit other than 1.

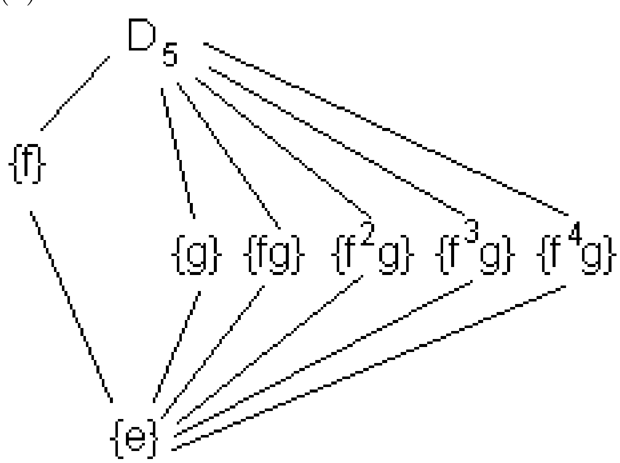
7. Let $R = \{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \ : \ a, b, c \in \mathbb{R}\}$, a subring of $M_2(\mathbb{R})$, and $I = \{ \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} \ : \ b, d \in \mathbb{R}\}$. Prove that I is an ideal in R, and $R/I \cong \mathbb{R}$ as rings. (Hint: Consider the function $\varphi : R \to \mathbb{R} : \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto a$.)

Solutions to Final Exam

1. (a) For $n = 1$, there is nothing to prove: $x * x = x * x$. Assume that $x^n * x = x * x^n$. Then $x^{n+1} * x = (x^n * x) * x = (x * x^n) * x = x * (x^n * x) = x * (x^{n+1})$. Therefore, $x^n * x = x * x^n$ for all positive integers n.
(b) $(1-1) - 1 = -1$, but $1 - (1-1) = 1$, i.e., 1 does not commute with $1 - 1$.

2. (a) 4, because $i^4 = 1$.
(b) ∞.
(c) Because 3 has order 2 in $\mathbb{Z}_6$ and 1 has order 4 in $\mathbb{Z}_4$, $(3, 1)$ has order $\gcd(2, 4) = 4$.
(d) All elements in a power set are their own inverses, so they all have order 2 except the identity, $\emptyset$.

3. (a) The factors of $|D_5| = 10$: 1, 2, 5, 10.
(b)



(c) $\{e\}$, D_5 and $\{f\}$ are normal; the two-element subgroups $\{g\}$, $\{fg\}$, $\{f^2g\}$, $\{f^3g\}$ and $\{f^4g\}$ are not normal. $\{f\}$ is normal because it is of index 2 in D_5. And $\{g\}$ is not normal because $fgf^{-1} = gf^4f^4 = gf^3 \notin \{g\}$.

4. Reflexive: Because the identity function i is in G, $i(x) = x$ for every x in $\{1, 2, \ldots, n\}$, so xRx. Symmetric: If xRy, then there is a function g in G for which $g(x) = y$; but then $x = g^{-1}(y)$, and $g^{-1} \in G$, so yRx. Transitive: If xRy and yRz, then there are functions g, h in G for which $g(x) = y$ and $h(y) = z$; and then $(h \circ g)(x) = h(g(x)) = h(y) = z$, and $h \circ g \in G$, so xRz.
(b) Clearly the identity function is in any G_x. Suppose $g, h \in G_x$, i.e., $g(x) = x$ and $h(x) = x$; then $(g \circ h)(x) = g(h(x)) = g(x) = x$, so $g \circ h \in G_x$. Suppose $g \in G_x$, i.e., $g(x) = x$; then $x = g^{-1}(x)$, so $g^{-1} \in G_x$.
(c) $g_1(x) = g_2(x)$ iff $g_2^{-1}(g_1(x)) = x$, iff $(g_2^{-1} \circ g_1)(x) = x$, iff $g_2^{-1} \circ g_1 \in G_x$, iff $g_1G_x = g_2G_x$.
(d) Because each image of x under an element of G corresponds to a left coset of G_x, according to (c), the number of images of x under elements of G is the number of left cosets of G_x in G, which is $[G : G_x] = |G|/|G_x|$, a factor of $|G|$.

5. (a) If there were an epimorphism φ from $\mathbb{Z}_{10}$ onto $\mathbb{Z}_4$, then $\mathbb{Z}_4 \cong \mathbb{Z}_{10}/\mathrm{Ker}(\varphi)$, so $4 = |\mathbb{Z}_4| = [\mathbb{Z}_{10} : \mathrm{Ker}(\varphi)]$, which is a factor of $|\mathbb{Z}_{10}| = 10$; but 4 is not a factor of 10.
(b) If there were an epimorphism φ from $\mathbb{Z}_12$ onto V, then $\mathrm{Ker}(\varphi)$ must have $|\mathbb{Z}_{12}|/|V| =$

$12/4 = 3$ elements, so $\text{Ker}(\varphi) = \langle 4 \rangle$. But in $\mathbb{Z}_{12}/\langle 4 \rangle$, $\langle 4 \rangle \oplus 1$ has order 4, while no element of V has order 4, so V is not isomorphic to $\mathbb{Z}_{12}/\langle 4 \rangle$. So there is no φ.

6. (a) 3, because $3 \odot 4 = 0$; (b) 6, because $6 \odot 6 = 0$; (c) 4, because $4 \odot 4 = 4$; and (d) 11, because $11 \odot 11 = 1$.

7. If the function φ described in the hint is a ring homomorphism, then I is clearly its kernel and $\mathbb{R}$ is its image, so the desired results will follow. So let's check that it is a homomorphism:

$$\varphi(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} + \begin{pmatrix} e & f \\ 0 & h \end{pmatrix}) = \varphi(\begin{pmatrix} a+e & b+f \\ 0 & d+h \end{pmatrix}) = a + e = \varphi(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}) + \varphi(\begin{pmatrix} e & f \\ 0 & h \end{pmatrix})$$

$$\varphi(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\begin{pmatrix} e & f \\ 0 & h \end{pmatrix}) = \varphi(\begin{pmatrix} ae & af+bh \\ 0 & dh \end{pmatrix}) = ae = \varphi(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix})\varphi(\begin{pmatrix} e & f \\ 0 & h \end{pmatrix})$$