Section 1: (Binary) Operations — Graphics

On a finite set, an operation can be defined by a table: On a set $S = \{a, b, c, d\}$,

$*$	a	b	c	d
a	a	b	c	d
b	d	c	b	a
c	c	c	d	d
d	d	d	c	c

means

$$\begin{array}{cccc} a*a=a & a*b=b & a*c=c & a*d=d \\ b*a=d & b*b=c & b*c=b & b*d=a \\ c*a=c & c*b=c & c*c=d & c*d=d \\ d*a=d & d*b=d & d*c=c & d*d=c \end{array}$$

For a fixed set X, and for A, B in $\mathcal{P}(X)$,

$$\begin{aligned} A \cup B &= \{x \in S : x \in A \text{ or } x \in B \text{ or both}\} \\ A \cap B &= \{x \in S : x \in A \text{ and } x \in B\} \\ A \triangle B &= \{x \in S : x \in A \text{ or } x \in B \text{ but not both}\} = (A \cup B) - (A \cap B) \end{aligned}$$

Example: $X = \{4, 7\}$, so $\mathcal{P}(X) = \{\emptyset, \{4\}, \{7\}, X\}$:

$\cup$	$\emptyset$	$\{4\}$	$\{7\}$	X
$\emptyset$	$\emptyset$	$\{4\}$	$\{7\}$	X
$\{4\}$	$\{4\}$	$\{4\}$	X	X
$\{7\}$	$\{7\}$	X	$\{7\}$	X
X	X	X	X	X

$\cap$	$\emptyset$	$\{4\}$	$\{7\}$	X
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{4\}$	$\emptyset$	$\{4\}$	$\emptyset$	$\{4\}$
$\{7\}$	$\emptyset$	$\emptyset$	$\{7\}$	$\{7\}$
X	$\emptyset$	$\{4\}$	$\{7\}$	X

$\triangle$	$\emptyset$	$\{4\}$	$\{7\}$	X
$\emptyset$	$\emptyset$	$\{4\}$	$\{7\}$	X
$\{4\}$	$\{4\}$	$\emptyset$	X	$\{7\}$
$\{7\}$	$\{7\}$	X	$\emptyset$	$\{4\}$
X	X	$\{7\}$	$\{4\}$	$\emptyset$

Polish notation: $+(a, b)$.
reverse Polish notations: $(a, b)+$
infix notation: $a + b$
juxtaposition: ab

Def: An operation $*$ on a set S is *commutative* iff, for every two elements a, b of S, $a * b = b * a$ (i.e., the function $*$ associates the ordered pairs (a, b) and (b, a) to the same element of S). And $*$ is *associative* iff, for all elements a, b, c of S, $(a * b) * c = a * (b * c)$.

Non-associative operations:

1. The cross-product of 3-vectors:

$$(\mathbf{i} \times \mathbf{i}) \times \mathbf{j} = \mathbf{0} \times \mathbf{j} = \mathbf{0} \qquad \text{but} \qquad \mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = \mathbf{i} \times \mathbf{k} = -\mathbf{j} \;.$$

2. Subtraction of real numbers: $(3 - 2) - 1 = 0$ but $3 - (2 - 1) = 2$.

Take two operations on the set $S = \{a, b, c\}$:

$*$	a	b	c
a	a	b	a
b	b	c	b
c	a	b	a

$\circ$	a	b	c
a	a	b	c
b	b	c	b
c	a	b	c

Of these, $\circ$ is not commutative ($a \circ c = c$ but $c \circ a = a$), while $*$ is commutative, by the symmetry of the table (though $*$ is not associative: $(b * b) * a = c * a = a$ but $b * (b * a) = b * b = c$.)

Associativity does hold "naturally" if the operation is itself, or is derived from, a function composition, because function compositions are clearly associative: $((f \circ g) \circ h)(x) = f(g(h(x))) = (f \circ (g \circ h))(x)$ — on both ends h is applied to x, then g is applied to $h(x)$, then f is applied to $g(h(x))$, so the results are identical.

Example: Matrix multiplication and linear transformations: We can check that every linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is given by a rule of the form $T(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$, so T is multiplication of each vector by a fixed matrix:

$$T(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = B \begin{pmatrix} x \\ y \end{pmatrix} \text{ , say, where } B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

In linear algebra, B was called the "matrix representation of T" (with respect to the standard basis). If A, C are the matrix representations of the linear transformations $S, U : \mathbb{R}^2 \to \mathbb{R}^2$, then for every $\begin{pmatrix} x \\ y \end{pmatrix}$ in $\mathbb{R}^2$,

$$\begin{aligned} ((AB)C) \begin{pmatrix} x \\ y \end{pmatrix} &= ((S \circ T) \circ U)(\begin{pmatrix} x \\ y \end{pmatrix}) \\ &= S(T(U(\begin{pmatrix} x \\ y \end{pmatrix}))) \\ &= (S \circ (T \circ U))(\begin{pmatrix} x \\ y \end{pmatrix}) \\ &= (A(BC)) \begin{pmatrix} x \\ y \end{pmatrix} , \end{aligned}$$

and because this works for every vector in $\mathbb{R}^2$, we get $(AB)C = A(BC)$. So matrix multiplication is associative *because* it reflects composition of linear transformations, which is "naturally" associative.

Def: If S is a set and $*$ is an associative operation on S, then the pair $(S, *)$ (or sometimes just S, if there is a natural choice for $*$) is called a *semigroup.*