Section 1: (Binary) Operations — Graphics

On a finite set, an operation can be defined by a table: On a set $S = \{a, b, c, d\}$,

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means

$a * a = a \quad a * b = b \quad a * c = c \quad a * d = d$

$b * a = d \quad b * b = c \quad b * c = b \quad b * d = a$

$c * a = c \quad c * b = c \quad c * c = d \quad c * d = d$

$d * a = d \quad d * b = d \quad d * c = c \quad d * d = c$
For a fixed set $X$, and for $A, B$ in $\mathcal{P}(X)$,

\begin{align*}
A \cup B &= \{ x \in S : x \in A \text{ or } x \in B \text{ or both} \} \\
A \cap B &= \{ x \in S : x \in A \text{ and } x \in B \} \\
A \triangle B &= \{ x \in S : x \in A \text{ or } x \in B \text{ but not both} \} = (A \cup B) - (A \cap B)
\end{align*}

Example: $X = \{4, 7\}$, so $\mathcal{P}(X) = \{\emptyset, \{4\}, \{7\}, X\}$:

\begin{center}
\begin{tabular}{c|cccc}
& $\emptyset$ & $\{4\}$ & $\{7\}$ & $X$ \\
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$\emptyset$ & $\emptyset$ & $\{4\}$ & $\{7\}$ & $X$ \\
$\{4\}$ & $\{4\}$ & $\{4\}$ & $X$ & $X$ \\
$\{7\}$ & $\{7\}$ & $X$ & $\{7\}$ & $X$ \\
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$\{7\}$ & $\{7\}$ & $X$ & $\emptyset$ & $\{4\}$ \\
$X$ & $X$ & $\{7\}$ & $\{4\}$ & $\emptyset$
\end{tabular}
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Polish notation: \(+ (a, b)\).
reverse Polish notations: \((a, b) +\)
infix notation: \(a + b\)
juxtaposition: \(ab\)
**Def:** An operation $\ast$ on a set $S$ is *commutative* iff, for every two elements $a, b$ of $S$, $a \ast b = b \ast a$ (i.e., the function $\ast$ associates the ordered pairs $(a, b)$ and $(b, a)$ to the same element of $S$). And $\ast$ is *associative* iff, for all elements $a, b, c$ of $S$, $(a \ast b) \ast c = a \ast (b \ast c)$.

Non-associative operations:
1. The cross-product of 3-vectors:

   \[
   (i \times i) \times j = 0 \times j = 0 \quad \text{but} \quad i \times (i \times j) = i \times k = -j.
   \]

2. Subtraction of real numbers: $(3 - 2) - 1 = 0$ but $3 - (2 - 1) = 2$.

Take two operations on the set $S = \{a, b, c\}$:

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Of these, $\circ$ is not commutative ($a \circ c = c$ but $c \circ a = a$), while $\ast$ is commutative, by the symmetry of the table (though $\ast$ is not associative: $(b \ast b) \ast a = c \ast a = a$ but $b \ast (b \ast a) = b \ast b = c$.)
Associativity does hold “naturally” if the operation is itself, or is derived from, a function composition, because function compositions are clearly associative: 
\[ ((f \circ g) \circ h)(x) = f(g(h(x))) = (f \circ (g \circ h))(x) \] — on both ends \( h \) is applied to \( x \), then \( g \) is applied to \( h(x) \), then \( f \) is applied to \( g(h(x)) \), so the results are identical.

Example: Matrix multiplication and linear transformations: We can check that every linear transformation \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) is given by a rule of the form 
\[ T\left( \begin{pmatrix} x \\ y \end{pmatrix} \right) = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}, \] so \( T \) is multiplication of each vector by a fixed matrix:
\[ T\left( \begin{pmatrix} x \\ y \end{pmatrix} \right) = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = B \begin{pmatrix} x \\ y \end{pmatrix}, \] say, where \( B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \).

In linear algebra, \( B \) was called the “matrix representation of \( T \)” (with respect to the standard basis). If \( A, C \) are the matrix representations of the linear transformations \( S, U : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \), then for every \( \begin{pmatrix} x \\ y \end{pmatrix} \) in \( \mathbb{R}^2 \),
\[
((AB)C)\begin{pmatrix} x \\ y \end{pmatrix} = ((S \circ T) \circ U)\begin{pmatrix} x \\ y \end{pmatrix} = S(T(U(\begin{pmatrix} x \\ y \end{pmatrix})))) = (S \circ (T \circ U))(\begin{pmatrix} x \\ y \end{pmatrix}) = (A(BC))\begin{pmatrix} x \\ y \end{pmatrix},
\]

and because this works for every vector in \( \mathbb{R}^2 \), we get \( (AB)C = A(BC) \). So matrix multiplication is associative because it reflects composition of linear transformations, which is “naturally” associative.
Def: If $S$ is a set and $*$ is an associative operation on $S$, then the pair $(S,*)$ (or sometimes just $S$, if there is a natural choice for $*$) is called a *semigroup*. 