3.2.5. Note first that there is a typo in this question: Rather than $V_\epsilon(x) \cap A = \{a\}$, it should say $V_\epsilon(a) \cap A = \{a\}$. Also, the relevant definitions here are Definitions 3.2.4 and 3.2.6, but as usual, I won't cite them (and I won't demand that you cite them) explicitly — I expect you to know what they say, but not what number our text gives them.

Now suppose first that a is an isolated point of A, i.e., it is in A but not a limit point of A. Because it is not a limit point, there is some ϵ-neighborhood $V_\epsilon(a)$ that does not intersect A in any other point than a — but that neighborhood <u>does</u> have a in it (at its center), so $V_\epsilon(a) \cap A = \{a\}$.

Conversely, suppose that there is an ϵ-neighborhood $V_\epsilon(a)$ such that $V_\epsilon(a) \cap A = \{a\}$. Then a is not a limit point of A, but we were told it was an element of A, so it is an isolated point of A.

3.2.10. (a) For each x in the universal set of which the E_λ's are all subsets:

$$\begin{aligned} x \in \left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^c &\iff x \notin \bigcup_{\lambda\in\Lambda} E_\lambda \\ &\iff x \notin E_\lambda \text{ for all } \lambda \in \Lambda \\ &\iff x \in E_\lambda^c \text{ for all } \lambda \in \Lambda \\ &\iff x \in \bigcap_{\lambda\in\Lambda} E_\lambda^c\ , \end{aligned}$$

so $\left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^c = \bigcap_{\lambda\in\Lambda} E_\lambda^c$. Similarly, for each x in the universal set,

$$\begin{aligned} x \in \left(\bigcap_{\lambda\in\Lambda} E_\lambda\right)^c &\iff x \notin \bigcap_{\lambda\in\Lambda} E_\lambda \\ &\iff x \notin E_\lambda \text{ for at least one } \lambda \in \Lambda \\ &\iff x \in E_\lambda^c \text{ for at least one } \lambda \in \Lambda \\ &\iff x \in \bigcup_{\lambda\in\Lambda} E_\lambda^c\ , \end{aligned}$$

so $\left(\bigcap_{\lambda\in\Lambda} E_\lambda\right)^c = \bigcup_{\lambda\in\Lambda} E_\lambda^c$.

(b) (i) Take a finite collection $\{E_\lambda\}_{\lambda\in\Lambda}$ of closed sets. Then each E_λ^c is an open set, so by Theorem 3.2.3(ii) their intersection is open. But by De Morgan's Law, that intersection, $\bigcap_{\lambda\in\Lambda} E_\lambda^c$, is equal to $\left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^c$, so by Theorem 3.2.13, the union $\bigcup_{\lambda\in\Lambda} E_\lambda$ is closed.
(ii) Take an arbitrary collection $\{E_\lambda\}_{\lambda\in\Lambda}$ of closed sets. Then each E_λ^c is an open set, so by Theorem 3.2.3(i) their union is open. But by De Morgan's Law, that union, $\bigcup_{\lambda\in\Lambda} E_\lambda^c$, is equal to $\left(\bigcap_{\lambda\in\Lambda} E_\lambda\right)^c$, so by Theorem 3.2.13, the intersection $\bigcap_{\lambda\in\Lambda} E_\lambda$ is closed.

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3.3.2. Suppose K is a a closed and bounded subset of $\mathbb{R}$, and let $(x_n)_{x\in\mathbb{N}}$ be a sequence in K; we must show that (x_n) has a subsequence that converges to an element of K: Because K is bounded, so is the sequence (x_n). By the Bolzano-Weierstrass Theorem (Theorem 2..5), (x_n) has a convergent subsequence (y_n), still in K. Because K is closed, the limit of (y_n) is also in K. Therefore, K is compact.

3.3.10. Suppose E is a "clompact" subset of $\mathbb{R}$. All one-element sets are closed, so one cover of E by closed sets is $\{\{x\} : x \in E\}$. Because E is clompact, there is a finite subset F of E for which $\{\{x\} : x \in F\}$ is a cover of E, i.e., the union of these one-element sets is E. But it is also the finite set F, so $E = F$. Thus, the only clompact subsets of $\mathbb{R}$ are the finite subsets.