

## Math 323 — Exam I

Make sure your reasoning is clear. Points are specified.

1. (20 points) Let  $A$  be a nonempty subset of  $\mathbb{R}$  that is bounded above. Then it can be shown (but don't do it!) that there is a sequence  $(a_n)$  of terms in  $A$  for which  $\lim a_n = \sup(A)$ .
  - (a) Give an example of a set  $A$  for which there is no sequence in  $A$  with limit  $\sup A$  that is eventually constant.
  - (b) Give an example of a set  $A$  for which every sequence in  $A$  with limit  $\sup A$  is eventually constant.
  - (c) Prove that if  $c$  is a negative real number, then the set  $cA = \{ca : a \in A\}$  is bounded below and  $\inf(cA) = c \cdot \sup(A)$ .
2. (25 points) Let us say that a sequence  $(c_n)_{n=1}^{\infty}$  of real numbers “*cervonges* to  $c$ ” (where  $c \in \mathbb{R}$ ) if and only if there is an  $N \in \mathbb{N}$  such that, for all  $n > N$  and all  $\varepsilon > 0$ ,  $|c_n - c| < \varepsilon$ .
  - (a) If a sequence  $(c_n)$  *cervonges* to  $c$ , does  $(c_n)$  converge to  $c$ ? Explain, and if not, give an example.
  - (b) If a sequence  $(c_n)$  converges to  $c$ , does  $(c_n)$  *cervonge* to  $c$ ? Explain, and if not, give an example.
3. (10 points) A problem in our text instructs us to define the phrase “converges to  $\infty$ ” — most texts would say “diverges to  $\infty$ ”. Define what it should mean to say “ $(a_n)$  diverges to  $-\infty$ ”. (Warning: “ $|a_n + \infty| < \varepsilon$ ” is gibberish. What should “close to  $-\infty$ ” mean?)
4. (20 points) Concerning the Algebraic Limit Theorem :
  - (a) Prove the product part of the ALT: If  $\lim a_n = a$  and  $\lim b_n = b$ , then  $\lim(a_n b_n) = ab$ . To eliminate a case in the proof, you may assume that  $a \neq 0$ .
  - (b) Using the ALT, prove that the limit of  $(3n + 1)/(2n - 5)$  is as expected.
5. (10 points) Prove that, if  $x_n \leq y_n \leq z_n$  for all  $n$  in  $\mathbb{N}$  and  $\lim x_n = \ell = \lim z_n$ , then  $\lim y_n$  also exists and is  $\ell$ . You may use without proof the fact that, if  $a \leq b \leq c$ , then  $|b| \leq \max(|a|, |c|)$ .
6. (15 points) True or false. If true, give a quick proof; if false, give a counterexample.
  - (a)  $\sup(AB) = (\sup A)(\sup B)$ . (Here,  $AB = \{ab : a \in A, b \in B\}$ .)
  - (b)  $\lim(a_n/b_n) = (\lim a_n)/(\lim b_n)$ .
  - (c) If  $a_n \leq b_n$  for all  $n$  in  $\mathbb{N}$ , and  $\lim a_n = a$  and  $\lim b_n = b$ , then  $a \leq b$ .

### Math 323 — Solutions to Exam I

- (a) One such  $A$  is the open interval  $(0, 1)$ , but any set that is bounded above and does not contain its sup would work.

(b) One such  $A$  is  $\{1\}$ , but any set in which the sup is in  $A$  but is separated from the rest of  $A$  by an open gap would work. (The technical phrase is that that sup is an “isolated point” of the set.)

(c) For each  $x$  in  $cA$ ,  $x = ca$  for some  $a$  in  $A$ , and  $\sup(A) \geq a$ , so  $c \cdot \sup(A) \leq ca = x$ ; thus  $x$  is bounded below by  $c \cdot \sup(A)$ . Now let  $b$  be a lower bound of  $cA$ ; then for all  $a$  in  $A$ , because  $ca \in cA$ ,  $b \leq ca$ , so  $b/c \geq a$ . Thus  $b/c$  is an upper bound for  $A$ , so  $b/c \geq \sup(A)$ , and hence  $b \leq c \cdot \sup(A)$ . It follows that  $c \cdot \sup(A)$  is the greatest lower bound of  $cA$ .
- (a) Yes. The definition of “cervonges” implies that  $c_n = c$  for all  $n \geq N$ ; so of course for any  $\varepsilon > 0$  there is an  $N$  for which  $|c_n - c| = 0 < \varepsilon$  for all  $n \geq N$ .

(b) No: The sequence  $(1/n)$  converges to 0, but it does not cervonge to 0.
- One version of the definition might be:  $a_n$  converges to  $-\infty$  iff, for each  $B > 0$ , there is an  $N \in \mathbb{N}$  for which, for all  $n \geq N$ ,  $a_n \leq -B$ .
- (a) Let  $\varepsilon > 0$  be given, and choose  $N$  in  $\mathbb{N}$  sufficiently large that for all  $n \geq N$ ,

$$|a_n - a| < \frac{\varepsilon}{2(|b| + 1)} \quad \text{and} \quad |b_n - b| < \frac{\varepsilon}{2|a|} \quad \text{and} \quad |b_n - b| < 1.$$

Then for such  $n$  we have  $||b_n| - |b|| < 1$ , so that  $|b_n| < |b| + 1$ , and so

$$\begin{aligned} |a_n b_n - ab| &= |(a_n - a)b_n + a(b_n - b)| \leq |a_n - a||b_n| + |a||b_n - b| \\ &< \frac{\varepsilon}{2(|b| + 1)}(|b| + 1) + |a|\frac{\varepsilon}{2|a|} = \varepsilon. \end{aligned}$$

Therefore,  $\lim(a_n b_n) = ab$ .

- (b) Because  $(3n + 1)/(2n - 5)$  is equal to  $(3 + \frac{1}{n})/(2 - \frac{5}{n})$ , the limit of the latter is equal to the limit of the former. But using the fact that  $\lim(c/n) = 0$  for any constant  $c$ , we get using the ALT that  $\lim(3 + \frac{1}{n}) = (\lim 3) + (\lim \frac{1}{n}) = 3 + 0 = 3$  and similarly  $\lim(2 - \frac{5}{n}) = (\lim 2) - (\lim \frac{5}{n}) = 2 - 0 = 2$ , so again by the ALT (and the fact that  $2 \neq 0$ ), we have  $\lim((3n + 1)/(2n - 5)) = 3/2$ .
- Let  $\varepsilon > 0$  be given, and pick  $N$  in  $\mathbb{N}$  so that, for all  $n \geq N$ ,  $|x_n - \ell| < \varepsilon$  and  $|z_n - \ell| < \varepsilon$ . Then because  $x_n - \ell \leq y_n - \ell \leq z_n - \ell$ , we have, for all  $n \geq N$ ,  $|y_n - \ell| \leq \max(|x_n - \ell|, |z_n - \ell|) < \varepsilon$ . Therefore  $\lim y_n = \ell$ .
- (a) False: A counterexample is  $A = \{-1\}$  and  $B = \{0, 1\}$ , because  $\sup(AB) = \sup\{-1, 0\} = 0$ , but  $(\sup A)(\sup B) = (-1)(1) = -1$ .

(b) False: The hypothesis that  $\lim b_n \neq 0$  is necessary. For instance  $a_n = 1/n$  and  $b_n = 1/n$  would make the left side equal to 1 and the right undefined.

(c) True, by the Order Limit Theorem.